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Solving An Optimal Control Problem With Free Final Time

S.Suvitha [1] and R.Geetharamani [2]

M. Phil,Department of Mathematics, KG College of Arts and Science, Coimbatore, India [1]

Assistant Professor, Department of Mathematics, KG College of Arts and Science, Coimbatore, India [2]

suvithasubramanie@gmail.com [1] and geetharamani.r@kgcas.com [2]

Abstract: **In this paper, we presents a Pontryagin Principle for Bolza problem. The procedure for solving a Bolza Problem is discussed. To solving an optimal control problem with free final time. The problem is to find an optimal control and optimal state.**

Keywords: Optimal Control, Bolza Problem, Pontryagin Principle Algorithm, Pontryagin Principle.

I. INTRODUCTION

The theory of optimal control has been developed for over forty years. With the advances of computer technique, optimal control is now widely used in multi-disciplinary applications such as biological systems, communications networks and socio-economic systems etc. As a result, more and more people will benefit greatly by learning to solve the optimal control problems numerically. An optimal control is a set of differential equation describing the paths of the control variables that minimize the cost function. The control theory [1] was the "Classic" book for studying the theory as well as many Problems (Observability, Controllability, Stable).In most books [3] [4] it is free final time problem that being tackled first to derive the necessary conditions for optimal control. In retrospect, [4] was the first and the "classic" book for studying the theory as well as many interesting cases. Necessary conditions for various systems were derived and explicit solutions were given when possible. Later [3] proved to be a concise yet excellent book with more engineering examples. One of the distinguish features of this book is that it introduced several iterative algorithms for solving problem numerically. Free final time problem [2] [5] were treated as an equivalent variation with one more state for time. However free final time problem general form and solve the problem by using algorithm to find an optimal control and optimal state.

II. BASIS DEFINITIONS

A. *OPTIMAL CONTROL*

Consider the linear varying system,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

and the cost functional,

$$J = \frac{1}{2}x^*(T)Fx(T) + \frac{1}{2}\int_0^T [x^*(t)Q(t)x(t) + u^*(t)R(t)u(t)]dt$$

Where $A(t), B(t), Q(t), R(t)$ are continuous on $[0,T]$,the terminal time T is specified,

F is a constant $n \times n$ symmetric positive semidefinite matrix,

$Q(t)$ is an $n \times n$ symmetric positive semidefinite matrix,

$R(t)$ is an $m \times m$ symmetric positive definite matrix and $u(t)$ is not constrained.

We shall show that the optimal control is a linear function of state,this is of the form

$$u(t) = G(t)x(t), \qquad t \in [0,T]$$

Where, $G(t)$ is an $m \times n$ matrix valued function.

B. *BOLZA PROBLEM*

Consider the optimal control system where the performance index is of general form containing a final (terminal) cost function in addition to the integral cost

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function. Such a optimal control problem is called the Bolza problem.

Given the system as,

$$\dot{x}(t) = f(x(t), u(t), t)$$

The performance index as

$$J = S(x(t_f), t_f) + \int_{t_0}^{t_f} V(x(t), u(t), t)\,dt,$$

and the boundary conditions as

$x(t_0) = x_0$ and t_f and $x(t_f) = x_f$ are free,

Where, x(t) and u(t) are n- and r-dimensional state and control vectors respectively.

III. PROCEDURE FOR SOLVING THE PONTRYAGIN PRINCIPLE FOR BOLZA PROBLEM

Step 1:

Form the pontryagin H function,

$$\mathrm{H}(x(t), u(t), \lambda(t), t) = V(x(t), u(t), t) + \lambda'(t) f(x(t), u(t), t)$$

Step 2:

Minimize H w.r.t. u(t)

$\left(\dfrac{\partial \mathrm{H}}{\partial u}\right)_* = 0$ and obtain $u^*(t) = h(x^*(t), \lambda^*(t), t)$.

Step 3:

Using the results of Step 2 in Step 1, find the optimal H^*

$$\mathrm{H}^*(x^*(t), h(x^*(t), \lambda^*(t), t), \lambda^*(t), t) = \mathrm{H}^*(x^*(t), \lambda^*(t), t).$$

Step 4:

Solve the set of 2n differential equations

$$\dot{x}^*(t) = +\left(\frac{\partial \mathrm{H}}{\partial \lambda}\right)_* \text{ and}$$

$$\dot{\lambda}^*(t) = -\left(\frac{\partial \mathrm{H}}{\partial x}\right)_*$$

With initial conditions x_0 and the final conditions

$$\left[\mathrm{H}^* + \frac{\partial S}{\partial t}\right]_{t_f} \delta t_f + \left[\left(\frac{\partial S}{\partial x}\right)_* - \lambda^*(t)\right]'_{t_f} \delta x_f = 0$$

Step 5:

Substitute the solutions of $x^*(t), \lambda^*$ (t) from step 4 into the expression for the optimal control $u^*(t)$ of step 2.

IV. EXAMPLE

Given a second order system as,

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -u(t) \qquad (1)$$

and the performance index as,

$$J = \frac{1}{2}\int_{t_0}^{t_f} u^2(t)\,dt \qquad (2)$$

Find the optimal control and optimal state, given the boundary conditions as,

$$x(0) = [1,2]^T;\ x(2) = [1,0]^T.$$

Assume that the control and state are unconstrained.

Solution:

We follow the step-by-step procedure given a algorithm. First by comparing the present system (1) and the performance index(2) with the general formulation of the Bolza Problem.

$$V(x(t), u(t), t) = v(u(t)) = \frac{1}{2}u^2(t)$$

$$f(x(t), u(t), t) = [f_1, f_2]^T$$

Where $f_1 = x_2(t),\ f_2 = -u(t)$

Step 1:

Form Hamiltonian function as

$$\mathrm{H} = \mathrm{H}(x_1(t), x_2(t), u(t), \lambda_1(t), \lambda_2(t)$$

$$= V(u(t) + \lambda'(t) f(x(t), u(t))$$

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$$=\frac{1}{2}u^2(t)+\lambda_1(t)x_2(t)+\lambda_2(t)(-u(t)) \quad (3)$$

Step 2:

To find $u^*(t)$ from,

$$\frac{\partial H}{\partial u}=0\rightarrow -u^*(t)+\lambda_2^*(t)=0\rightarrow u^*(t)=\lambda_2^*(t) \quad (4)$$

Step 3:

Using the results of Step 2 in Step 1,find the optimal H^* as

$$H^*(x_1^*(t),x_2^*(t),\lambda_1^*(t),\lambda_2^*(t))=\frac{1}{2}\lambda_2^{*^2}(t)+\lambda_1^*(t)x_2^*(t)+\lambda_2^{*^2}(t)$$

$$=\lambda_1^*(t)x_2^*(t)+\frac{3}{2}\lambda_2^{*^2}(t) \quad (5)$$

Step 4:

Obtain the state and costate equations from,

$$\dot{x}_1^*(t)=+\left(\frac{\partial H}{\partial \lambda_1}\right)=x_2^*(t)$$

$$\dot{x}_2^*(t)=+\left(\frac{\partial H}{\partial \lambda_2}\right)=\frac{3}{2}\lambda_2^*(t)$$

$$\dot{\lambda}_1^*(t)=-\left(\frac{\partial H}{\partial x_1}\right)=0$$

$$\dot{\lambda}_2^*(t)=-\left(\frac{\partial H}{\partial x_2}\right)=-\lambda_1^*(t)$$

Solving the previous equations, we have the optimal state and costate as,

Integrate the above equations we get,

$$\dot{\lambda}_2^*(t)=-\int\lambda_1^*(t)$$

$$\lambda_2^*(t)=-\lambda_1^*(t)+c_4$$

$$\dot{\lambda}_1^*(t)=0$$

$$\lambda_1^*(t)=c_3 \quad (6)$$

$$\lambda_2^*(t)=-c_3(t)+c_4 \quad (7)$$

$$\dot{x}_2^*(t)=\frac{3}{2}\lambda_2^*(t)$$

$$x_2^*(t)=-\frac{3}{4}t^2c_3+\frac{3}{2}c_4t+c_2 \quad (8)$$

$$\dot{x}_1^*(t)=x_2^*(t)$$

$$x_1^*(t)=-\frac{1}{4}t^3c_3+\frac{3}{4}t^2c_4+c_2t+c_1 \quad (9)$$

Step 5:

Obtain the optimal control from

$$u^*(t)=\lambda_2^*(t)$$

$$u^*(t)=-c_3t+c_4$$

Where, c_1,c_2,c_3 and c_4 are constants evaluated using the given boundary conditions. Therefore the values are,

$$x_1(0)=1\Rightarrow x(0)=1$$

$$c_1=1$$

$$x_2(0)=2\Rightarrow x(0)=2$$

$$c_2=2$$

$$x_1(2)=1\Rightarrow x(2)=1$$

$$-4=-2c_3+3c_4 \quad (8)$$

$$x_2(2)\Rightarrow 0\Rightarrow x(2)=0$$

$$-2=-3c_3+3c_4 \quad (9)$$

Solving above equations we get,

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$$c_3 = -2$$

$$c_4 = \frac{-8}{3}$$

Finally, we have the optimal states, costates and control as,

$$\lambda_1^*(t) = -2$$

$$\lambda_2^*(t) = 2t - 2.6$$

$$x_1^*(t) = 0.5t^3 - 2t^2 + 2t + 1$$

$$x_2^*(t) = 1.5t^2 - 4t + 2$$

$$u^*(t) = 2t - 2.6$$

V. CONCLUSION

The process of solve an optimal control problem has been completed. The results of optimal states and optimal controls are obtained by using the procedure of Pontryagin Principle for Bolza Problem.

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