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CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS INVOLVING q-DERIVATIVE OPERATOR

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ABSTRACT

q-derivative operator has wide range of application in mathematics as well as in physics. Recently, we can see many papers in the area of Geometric function theory also. In this paper, we introduce and estimating first two MacLaurin coefficients for new subclasses of analytic and bi-univalent functions with respect to q-derivative operator. Moreover we derive another subclass of analytic and bi-univalent functions as a special consequences of this results.

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1 Introduction and preliminaries

Let $\mathcal{A}$ be the class of functions f which are analytic in the open unit disc $\mathbb{U}=\{z: z\in\mathbb{C}: |z|<1\}$ and is of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \qquad (z\in\mathbb{U}). \tag{1.1}$$

The well-known Koebe one-quarter theorem[5] ensures that the image of $\mathbb{U}$ under every univalent function $f\in\mathcal{A}$ contains a disk of radius 1/4. Hence every univalent function f has an inverse f^{-1} satisfying $f^{-1}(f(z))=z,(z\in\mathbb{U})$ and

$$f^{-1}(f(w))=w,\left(|w|<r_0(f), r_0(f)\geq 1/4\right)$$

where,

$$g(w)=f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots \tag{1.2}$$

A function $f\in\mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For example, functions in the class Σ are given below[20]:

$$\frac{z}{1-z}, \quad -log(1-z), \quad \frac{1}{2}log\left(\frac{1+z}{1-z}\right).$$

In 1967, Lewin[14] introduced the class Σ of bi-univalent functions and shown that $|a_2|<1.51$. In 1969, Netanyahu[16] showed that $max_{f\in\Sigma}|a_2|=4/3$ and Suffridge[22] have given an example of $f\in\Sigma$ for which

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$|a_2| = 4/3$. Later, in 1980, Brannan and Clunie[3] improved the result as $|a_2| \le \sqrt{2}$. In 1985, Kedzier-awski[12] proved this conjecture for a special case when the function f and f^{-1} are starlike. In 1984, Tan[23] proved that $|a_2| \le 1.485$ which is the best estimate for the function in the class of bi-univalent functions.

Recently, many authors have introduced and studied various subclasses of analytic and bi-univalent functions. Some of the recent analysis in this topics are [8, 9, 21, 24] for reference to the readers. Brannan and Taha[4] introduced certain subclasses of the bi-univalent function class Σ for the familiar subclasses $S^*(\alpha)$ and $C(\alpha)$. Ali et al.[1] widen the result of Brannan and Taha using subordination.

The concept of q-analog was first introduced by Jackson[10]. Mohammed and Darus studied the geometric analog of some subclasses of analytic functions by means of the q-difference operator $D_q f(z)$ for $0 < q < 1$, [15].

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \quad (z \neq 0) \tag{1.3}$$

$D_q f(0) = f'(0)$ *and* $D_q^2 f(z) = D_q(D_q f(z))$. From (2), we deduce that,

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1} \tag{1.4}$$

where
$$[k]_q = \frac{1-q^k}{1-q}. \tag{1.5}$$

AS $q \to 1^-, [k]_q \to k.$ For a function $h(z) = z^k$, we observe that ,

$$D_q(h(z)) = D_q(z^k) = \frac{1-q^k}{1-q} z^{k-1} = [k]_q z^{k-1},$$

$$\lim_{q\to 1} (D_q(h(z))) = \lim_{q\to 1} ([k]_q z^{k-1}) = kz^{k-1} = h'(z),$$

where h' is the ordinary derivative.

As a right inverse, Jackson[10, 11] introduced the q-integral

$$\int_0^z f(t) d_q t = z(1-q) \sum_{k=0}^{\infty} q^k f(zq^k),$$

provided that the series converges. For a function $h(z) = z^k$, we have

$$\int_0^z h(t) d_q t = \int_0^z t^k d_q t = \frac{z^{k+1}}{[k+1]_q} \quad (k \neq -1),$$

$$\lim_{q\to 1^-} \int_0^z h(t) d_q t = \lim_{q\to 1^-} \frac{z^{k+1}}{[k+1]_q} = \frac{z^{k+1}}{k+1} = \int_0^z h(t) dt,$$

where $\int_0^z h(t)dt$ is the ordinary integral. Note that the q-difference operator plays an important role in the theory of hypergeometric series and quantum physics (see for instance[2, 6, 7, 13, 19]). One can clearly see that $D_q f(z) \to f'(z)$ as $q \to 1^-$. This difference operator helps us to generalize the classes of starlike and convex

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functions S^* analytically.

K.S Padmanabhan and R. Parvatham[18] introduced and studied the class of functions $\mathcal{P}_m(\beta)$ for $m \geq 2$ and $0 \leq \beta < 1$, denote the class of analytic univalent functions p in $\mathcal{U}$ with the normalization $p(0) = 1$ and satisfying the conditions

$$\int_0^{2\pi} \left| \frac{Re\, p(z) - \beta}{1 - \beta} \right| d\theta < m\pi, \quad for \quad m \geq 2.$$

For $\beta = 0$, we can write $\mathcal{P}_m =: \mathcal{P}_m(0)$. Paatero[17] proved that every functions $p \in \mathcal{P}_m$ can be written by the Stieltjes integral representation

$$p(z) = \int_0^{2\pi} \frac{1 + ze^{it}}{1 + ze^{it}} d\mu(t), \tag{1.6}$$

where $\mu(t)$ is a real valued function with bounded variation on $[0, 2\pi]$ which satisfies

$$\int_0^{2\pi} d\mu(t) = 2\pi \quad and \quad \int_0^{2\pi} |\, d\mu(t)| \leq m\pi, \quad m \geq 2. \tag{1.7}$$

With these brief introduction, we now define the two new subclass of function class Σ and finding the coefficient estimates with the help of q-derivative operator.

Definition 1 *For* $\gamma \in \mathbb{C}, 0 \leq \beta < 1$ *and* $m \geq 2$, *suppose a function* $f \in \mathcal{A}$ *is said to be in the class* $S^*_{q,\Sigma}(\gamma, m, \beta)$ *if it satisfies the condition* $1 + \frac{1}{\gamma}\left[\frac{zD_q f(z)}{f(z)} - 1\right] \in \mathcal{P}_m(\beta)$,

and

$$1 + \frac{1}{\gamma}\left[\frac{wD_q g(w)}{g(w)} - 1\right] \in \mathcal{P}_m(\beta),$$

where g is defined in (??) and $z, w \in \mathcal{U}$.

Definition 2 *For* $\gamma \in \mathbb{C}, 0 \leq \beta < 1$ *and* $m \geq 2$, *suppose a function* $f \in \mathcal{A}$ *is said to be in the class* $C_{q,\Sigma}(\gamma, m, \beta)$ *if it satisfies the condition*

$$1 + \frac{1}{\gamma}\left[\frac{zD_q(D_q f(z))}{D_q f(z)}\right] \in \mathcal{P}_m(\beta),$$

and

$$1 + \frac{1}{\gamma}\left[\frac{wD_q(D_q g(w))}{D_q g(w)}\right] \in \mathcal{P}_m(\beta),$$

where g is defined in (??) and $z, w \in \mathcal{U}$.

The main object of this paper is to find the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclass of the function class Σ.

These two subclasses may also reduce to many of the new subclasses for different choices of parameters $\gamma, q,$ and β. We can see the results of these reduced subclass as corollaries.

2 main results

In order to prove our main results we need the following Lemma:

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Lemma 1 *Let* $\Phi(z)=1+h_1z+h_2z^2+\cdots$, $z\in U$, *such that* $\Phi\in P_m(\beta)$ *having the normalization* $\Phi(0)=1$ *then*

$$|h_n|\leq m(1-\beta),\quad n\geq 1.$$

Theorem 1 *If the function* f *given by (1.1) be in the class* $S^*_{q,\Sigma}(\gamma,m,\beta)$ *then*

$$|a_2|\leq min\left\{\sqrt{\frac{|\gamma|m(1-\beta)}{([3]_q-[2]_q)}},\frac{|\gamma|m(1-\beta)}{([2]_q-1)}\right\},$$

and

$$|a_3|\leq|\gamma|m(1-\beta)\left[\frac{1}{([3]_q-1)}+\frac{m(1-\beta)}{([2]_q-1)^2}\right].$$

Proof. Since $f\in S^*_{q,\Sigma}(\gamma,m,\beta)$ and $g=f^{-1}$, then there exists the functions ϕ,ψ having the Taylor series expansion

$$\phi(z)=1+c_1z+c_2z^2+\cdots,\quad z\in U, \tag{2.1}$$

and

$$\psi(w)=1+d_1w+d_2w^2+\cdots,\quad w\in U. \tag{2.2}$$

Now Definition 1 satisfying

$$1+\frac{1}{\gamma}\left[\frac{zD_qf(z)}{f(z)}-1\right]=\phi(z),$$

and

$$1+\frac{1}{\gamma}\left[\frac{wD_qg(w)}{g(w)}-1\right]=\psi(w).$$

By simple computation, we have

$$1+\frac{1}{\gamma}\left[\frac{zD_qf(z)}{f(z)}-1\right]=1+\frac{1}{\gamma}\left\{([2]_q-1)a_2z+\left[([3]_q-1)a_3-([2]_q-1)a_2^2\right]z^2+\cdots\right\}, \tag{2.3}$$

and

$$1+\frac{1}{\gamma}\left[\frac{wD_qg(w)}{g(w)}-1\right]=1+\frac{1}{\gamma}\left\{-([2]_q-1)a_2w+\left[(1-[3]_q)a_3+(2[3]_q-[2]_q-1)a_2^2\right]w^2-\cdots\right\}. \tag{2.4}$$

On equating the like powers of z and w from equations (??),(??),(??) and (??) we get

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$$\frac{1}{\gamma}([2]_q - 1)a_2 = c_1, \tag{2.5}$$

$$\frac{1}{\gamma}\left[([3]_q - 1)a_3 - ([2]_q - 1)a_2^2\right] = c_2, \tag{2.6}$$

$$-\frac{1}{\gamma}([2]_q - 1)a_2 = d_1, \tag{2.7}$$

and

$$\frac{1}{\gamma}\left[(1 - [3]_q)a_3 + (2[3]_q - [2]_q - 1)a_2^2\right] = d_2. \tag{2.8}$$

Since ϕ and ψ clearly satisfy the condition of Lemma 1 we have

$$|c_n| \leq m(1-\beta), \tag{2.9}$$

$$|d_n| \leq m(1-\beta), \tag{2.10}$$

for every $n \geq 1$. Now considering (??) and (??), we obtain

$$c_1 = -d_1. \tag{2.11}$$

Also from (??), (??), (??) and (??)

$$|a_2|^2 \leq \frac{|\gamma|}{2}\frac{(|c_2| + |d_2|)}{([3]_q - [2]_q)} \leq \frac{|\gamma| m(1-\beta)}{([3]_q - [2]_q)}, \tag{2.12}$$

this yields,

$$|a_2| \leq \sqrt{\frac{|\gamma| m(1-\beta)}{([3]_q - [2]_q)}}. \tag{2.13}$$

On the other hand from (??) and (??), we get

$$|a_2| \leq \frac{|\gamma| m(1-\beta)}{([2]_q - 1)}. \tag{2.14}$$

Hence equations (??) and (??) gives the estimates of $|a_2|$.

Next to find the bounds on $|a_3|$, by further computations from (??), (??) and (??), we can easily get

$$|a_3| \leq |\gamma| m(1-\beta)\left[\frac{1}{([3]_q - 1)} + \frac{|\gamma| m(1-\beta)}{([2]_q - 1)^2}\right], \tag{2.15}$$

which completes the proof of Theorem 1.

We obtain the following corollary by setting $\gamma = 1$.

Corollary 1 *Let* $f \in \mathcal{A}$ *given by (1.1) be in the class* $S^*_{q,\Sigma}(1, m, \beta)$, *then*

$$|a_2| \leq min\left\{\sqrt{\frac{m(1-\beta)}{([3]_q - [2]_q)}}, \frac{m(1-\beta)}{([2]_q - 1)}\right\},$$

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and $$|a_3|\le m(1-\beta)\left[\frac{1}{([3]_q-1)}+\frac{m(1-\beta)}{([2]_q-1)^2}\right].$$

Setting $q\to 1^-$, we get the following corollary.

Corollary 2 *Let* $f\in\mathcal{A}$ *given by (1.1) be in the class* $S^*_\Sigma(\gamma,m,\beta)$, *then*

$$\begin{aligned}|a_2| &\le min\left\{\sqrt{|\gamma|m(1-\beta)},|\gamma|m(1-\beta)\right\}\\ &\le\sqrt{|\gamma|m(1-\beta)},\end{aligned}$$

and $$|a_3|\le|\gamma|m(1-\beta)\left[\frac{1}{2}+m(1-\beta)\right].$$

Setting $\beta=0$ in Theorem 1, we have

Corollary 3 *Let* $f\in\mathcal{A}$ *given by (1.1) be in the class* $S^*_{q,\Sigma}(\gamma,m)$, *then*

$$|a_2|\le min\left\{\sqrt{\frac{|\gamma|m}{([3]_q-[2]_q)}},\frac{|\gamma|m}{([2]_q-1)}\right\},$$

$$|a_3|\le|\gamma|m\left[\frac{1}{([3]_q-1)}+\frac{m}{([2]_q-1)^2}\right].$$

If $\gamma=1$ and $q\to 1^-$, then we have the following corollary.

Corollary 4 *Let* $f\in\mathcal{A}$ *given by (1.1) be in the class* $S^*_\Sigma(1,m,\beta)$, *then*

$$\begin{aligned}|a_2| &\le min\left\{\sqrt{m(1-\beta)},m(1-\beta)\right\}\\ &\le\sqrt{m(1-\beta)},\end{aligned}$$

and $$|a_3|\le m(1-\beta)\left[\frac{1}{2}+m(1-\beta)\right].$$

By setting $\beta=0$ and $\gamma=1$ in Theorem 1, we have obtain the following result.

Corollary 5 *Let* $f\in\mathcal{A}$ *given by (1.1) be in the class* $S^*_{q,\Sigma}(1,m)$, *then*

$$|a_2|\le min\left\{\sqrt{\frac{m}{([3]_q-[2]_q)}},\frac{m}{([2]_q-1)}\right\},$$

and

$$|a_3|\le m\left[\frac{1}{([3]_q-1)}+\frac{m}{([2]_q-1)^2}\right].$$

Choosing $\beta=0,\gamma=1$ and $q\to 1^-$, in Theorem 1, we have

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Corollary 6 *Let* $f \in A$ *given by (1.1) be in the class* $S^*_\Sigma(1,m)$, *then*

$$|a_2| \le min\{\sqrt{m}, m\} \le \sqrt{m},$$

and

$$|a_3| \le m\left[\frac{1}{2}+m\right].$$

Theorem 2 *If the function* f *given by (1.1) be in the class* $C_{q,\Sigma}(\gamma,m,\beta)$ *then*

$$|a_2| \le min\left\{\sqrt{\frac{|\gamma| m(1-\beta)}{[2]_q([3]_q-[2]_q)}}, \frac{|\gamma| m(1-\beta)}{[2]_q}\right\},$$

and $$|a_3| \le |\gamma| m(1-\beta)\left[\frac{1}{([2]_q[3]_q)}+\frac{m(1-\beta)}{[2]_q^2}\right].$$

Proof. Since $f \in S^*_{q,\Sigma}(\gamma,m,\beta)$ and $g = f^{-1}$, then

$$1+\frac{1}{\gamma}\left[\frac{zD_q(D_q f(z))}{D_q f(z)}\right] = 1+\frac{1}{\gamma}\left\{[2]_q a_2 z+[2]_q\left([3]_q a_3-[2]_q a_2^2\right)z^2+\cdots\right\}, \tag{2.16}$$

$$1+\frac{1}{\gamma}\left[\frac{wD_q(D_q g(w))}{D_q g(w)}\right] = 1+\frac{1}{\gamma}\left\{-[2]_q a_2 w+[-[2]_q[3]_q a_3+\left(2[2]_q[3]_q-[2]_q^2\right)a_2^2]w^2-\cdots\right\}. \tag{2.17}$$

Now using the same procedure as in Theorem 1 we get the desired results of $|a_2|$ and $|a_3|$.

This completes the proof of Theorem 2.

Let $\gamma = 1$ and $f \in A$ given by (1.1) be in the class $C_{q,\Sigma}(m,\beta)$ we have obtain the following corollary.

Corollary 7 *Let* $f \in A$ *given by (1.1) be in the class* $C_{q,\Sigma}(1,m,\beta)$, *then*

$$|a_2| \le min\left\{\sqrt{\frac{m(1-\beta)}{[2]_q([3]_q-[2]_q)}}, \frac{m(1-\beta)}{[2]_q}\right\},$$

and

$$|a_3| \le m(1-\beta)\left[\frac{1}{([2]_q[3]_q)}+\frac{m(1-\beta)}{[2]_q^2}\right].$$

Setting $q \to 1^-$ in Theorem 2, we get the result as follows.

Corollary 8 *Let* $f \in A$ *given by (1.1) be in the class* $C_\Sigma(\gamma,m,\beta)$, *then*

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$$|a_2| \quad \leq min\left\{\sqrt{\frac{|\gamma|m(1-\beta)}{2}}, \frac{|\gamma|m(1-\beta)}{2}\right\}$$

$$\leq \sqrt{\frac{|\gamma|m(1-\beta)}{2}},$$

and

$$|a_3| \leq |\gamma| m(1-\beta)\left[\frac{1}{6}+\frac{m(1-\beta)}{4}\right].$$

Setting $\beta = 0$ in Theorem 2, we obtain the following corollary.

Corollary 9 *Let* $f \in A$ *given by (1.1) be in the class* $C_{q,\Sigma}(\gamma, m)$, *then*

$$|a_2| \leq min\left\{\sqrt{\frac{|\gamma|m}{[2]_q([3]_q-[2]_q)}}, \frac{|\gamma|m}{[2]_q}\right\},$$

and

$$|a_3| \leq |\gamma| m\left[\frac{1}{([2]_q[3]_q)}+\frac{m}{[2]_q^2}\right].$$

Let $\beta = 0$ and $q \to 1^-$ in Theorem 2, we have

Corollary 10 *Let* $f \in A$ *given by (1.1) be in the class* $C_{\Sigma}(\gamma, m)$, *then*

$$|a_2| \quad \leq min\left\{\sqrt{\frac{|\gamma|m}{2}}, \frac{|\gamma|m}{2}\right\}$$

$$\leq \sqrt{\frac{|\gamma|m}{2}},$$

$$|a_3| \leq |\gamma| m\left[\frac{1}{6}+\frac{m}{4}\right].$$

If $\beta = 0$ and $\gamma = 1$ then we have the following corollary.

Corollary 11 *Let* $f \in A$ *given by (1.1) be in the class* $C_{q,\Sigma}(1, m)$, *then*

$$|a_2| \leq min\left\{\sqrt{\frac{m}{[2]_q([3]_q-[2]_q)}}, \frac{m}{[2]_q}\right\},$$

and

$$|a_3| \leq m\left[\frac{1}{([2]_q[3]_q)}+\frac{m}{[2]_q^2}\right].$$

Setting $\beta = 0; \gamma = 1$ and $q \to 1^-$ in Theorem 2, we have the following corollary.

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Corollary 12 *Let* $f \in \mathcal{A}$ *given by (1.1) be in the class* $C_\Sigma(1,m)$, *then*

$$|a_2| \le min\left\{\sqrt{\frac{m}{2}}, \frac{m}{2}\right\} \le \sqrt{\frac{m}{2}},$$

and

$$|a_3| \le m\left[\frac{1}{6} + \frac{m}{4}\right].$$

As a special consequences of our results, we now define the following:

Definition 3 *For* $0 < q < 1, 0 \le \alpha < 1, 0 \le \beta < 1$, *a function* $f \in \mathcal{A}$ *is said to be in the class* $\mathcal{M}_{q,\Sigma}(k,\beta,\alpha)$ *if it satisfy the following conditions*

$$(1-\alpha)\frac{zD_q f(z)}{f(z)} + \alpha\left(1 + \frac{zD_q(D_q f(z))}{D_q f(z)}\right) \in \mathcal{P}_m(\beta), \quad m \ge 2$$

and

$$(1-\alpha)\frac{wD_q g(w)}{g(w)} + \alpha\left(1 + \frac{wD_q(D_q g(w))}{D_q g(w)}\right) \in \mathcal{P}_m(\beta), \quad m \ge 2$$

where $g = f^{-1}$ defined in (??) and $z, w \in \mathcal{U}$.

For the function $f \in \mathcal{A}$ be in the class $\mathcal{M}_{q,\Sigma}(m,\beta,\alpha)$, the following estimation holds.

Theorem 3 *If the function* f *given by (1.1) be in the class* $\mathcal{M}_{q,\Sigma}(m,\beta,\alpha)$ *then*

$$|a_2| \le min\left\{\sqrt{\frac{m(1-\beta)}{[(1-\alpha)+[2]_q\alpha]([3]_q-[2]_q)}}, \frac{m(1-\beta)}{[([2]_q-1)+\alpha]}\right\},$$

and

$$|a_3| \le m(1-\beta)\left\{\frac{1}{\left[(1-\alpha)([3]_q-1)+\alpha[2]_q[3]_q\right]} + \frac{m(1-\beta)}{\left[([2]_q-1)+\alpha\right]^2}\right\}.$$

Proof. Since $f \in \mathcal{M}_{q,\Sigma}(m,\beta,\alpha)$ and $g = f^{-1}$, consider the functions ϕ, ψ with $\phi(0) = 1$ and $\phi(0) = 1$ satisfying the conditions

$$(1-\alpha)\frac{zD_q f(z)}{f(z)} + \alpha\left(1 + \frac{zD_q(D_q f(z))}{D_q f(z)}\right) = \phi(z),$$

and

$$(1-\alpha)\frac{wD_q g(w)}{g(w)} + \alpha\left(1 + \frac{wD_q(D_q g(w))}{D_q g(w)}\right) = \psi(w).$$

Now simple calculation yields,

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$$(1-\alpha)\frac{zD_q f(z)}{f(z)} + \alpha\left(1+\frac{zD_q(D_q f(z))}{D_q f(z)}\right) = 1+[([2]_q-1)+\alpha]a_2 z + \left\{\left[(1-\alpha)([3]_q-1)+\alpha[2]_q[3]_q\right]a_3 - \left[(1-\alpha)([2]_q-1)+\alpha[2]_q^2\right]a_2^2\right\}z^2 + \dots, \tag{2.18}$$

and

$$(1-\alpha)\frac{wD_q g(w)}{g(w)} + \alpha\left(1+\frac{wD_q(D_q g(w))}{D_q g(w)}\right) = 1-[([2]_q-1)+\alpha]a_2 w + \left\{\left[(1-\alpha)(1-[3]_q)-\alpha[2]_q[3]_q\right]a_3 + \left[(1-\alpha)(2[3]_q-[2]_q-1)-\alpha[2]_q^2+2\alpha[2]_q[3]_q\right]a_2^2\right\}w^2 - \dots \tag{2.19}$$

On equating the like powers of z and w from equations (??),(??),(??) and (??) we have

$$[([2]_q-1)+\alpha]a_2 = c_1, \tag{2.20}$$

$$\left[(1-\alpha)(1-[3]_q)-\alpha[2]_q[3]_q\right]a_3 + \left[(1-\alpha)(2[3]_q-[2]_q-1)-\alpha[2]_q^2+2\alpha[2]_q[3]_q\right]a_2^2 = c_2, \tag{2.21}$$

$$-[([2]_q-1)+\alpha]a_2 = d_1, \tag{2.22}$$

and

$$\left[(1-\alpha)(1-[3]_q)-\alpha[2]_q[3]_q\right]a_3 + \left[(1-\alpha)(2[3]_q-[2]_q-1)-\alpha[2]_q^2+2\alpha[2]_q[3]_q\right]a_2^2 = d_2. \tag{2.23}$$

Now considering (??) and (??), we obtain

$$c_1 = -d_1. \tag{2.24}$$

Adding (??) and (??), and from the inequalities (??) and (??) we can reduce,

$$|a_2| \le \sqrt{\frac{m(1-\beta)}{[(1-\alpha)+[2]_q\alpha]([3]_q-[2]_q)}}, \tag{2.25}$$

also from (??) and (??), we have

$$|a_2| \le \frac{m(1-\beta)}{[([2]_q-1)+\alpha]} \tag{2.26}$$

Hence equations (??) and (??) gives the estimates of $|a_2|$.

In order to find the bounds on $|a_3|$, we can make use of equations (??), (??) and (??), which gives

$$|a_3| \le m(1-\beta)\left\{\frac{1}{\left[(1-\alpha)([3]_q-1)+\alpha[2]_q[3]_q\right]} + \frac{m(1-\beta)}{\left[([2]_q-1)+\alpha\right]^2}\right\}, \tag{2.27}$$

this completes the proof of Theorem 3.

Remark 1 *If* $\alpha = 0$, *then the function* f *be in the class* $\mathcal{M}_{q,\Sigma}(m,\beta,\alpha)$ *reduces to the subclass* $S^*_{q,\Sigma}(m,\beta)$ *defined in Corollary 1.*

Remark 2 *If* $\alpha = 1$, *then the function* f *be in the class* $\mathcal{M}_{q,\Sigma}(m,\beta,1)$ *reduces to the subclass*

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$C_{q,\Sigma}(m,\beta)$ *defined in Corollary 7.*

Setting $\beta=0$ and $q\to 1^{-}$ then we get the following corollary.

Corollary 13 *If* $f\in \mathrm{A}$ *be in the class* $\mathrm{M}_{\Sigma}(m,\alpha)$ *then*

$$|a_2|\le min\left\{\sqrt{\frac{m}{(1+\alpha)}},\frac{m}{1+\alpha}\right\}\le\sqrt{\frac{m}{(1+\alpha)}},$$

and

$$|a_3|\le m\left\{\frac{1}{2(1-\alpha)+6\alpha}+\frac{m}{(1+\alpha)^2}\right\}.$$

References

[1] R. M. Ali, L. S. Keong, V. Ravichandran, Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions. Appl. Math. Lett. 25 (2012), no. 3, 344-351.

[2] G. E. Andrews, Applications of basic hypergeometric functions, SIAM Rev. **16** (1974), 441-484.

ISSN2394-3777 (Print)
ISSN2394-3785 (Online)
Available online atwww.ijartet.com

International Journal of Advanced Research Trends in Engineering and Technology (IJARTET)
Vol. 5, Special Issue 9, March 2018

[3] D. A. Brannan, J. Clunie, Aspects of contemporary complex analysis, Academic Press, New York Londan, 1980.

[4] D. A. Brannan, T. S. Taha, On some classes of bi-univalent functions, Studia Univ. Babes, Bolyai Math. 31 (1986), n0. 2, 70-77.

[5] P. L. Duren, Univalent functions, Grundlehren der Mathematischen Wissenscafeten, 259, Spinger, New York, 1983.

[6] T. Ernst, The History of q-calculus and a New Method, Licentiate Dissertation, Uppsala, 2001.

[7] N. J. Fine, Basic hypergeometric series and applications, Mathematical Surveys and Monographs, 27, Amer. Math. Soc., Providence, RI, 1988.

[8] B. A. Frasin, Coefficient bounds for certain classes of bi-univalent functions, Hacet. J. Math. Stat, 43(3)(2014), 383-389.

[9] T. Hayami and S. Owa, Coefficient bounds for bi-univalent functions, Pan Amer. Math. J., 22(4) (2012), 15-26.

[10] F. H. Jackson, On q-finctions and a certain difference operator, Transactions of the Royal Society of Edinburgh, 46(1908) 253–281.

[11] F. H. Jackson, On q-definite integrals, Quarterly J.Pure Appl. Math., 41(1910) 193-203.

[12] A. Kedzierawski, J. Waniurski, Bi-univalent polynomials of small degree. Complex Variables Theory Appl. 10(1988), no. 2-3, 97-100.

[13] Kirillov, A. N., Dilograithm identities, Progr. Theoret. Phys.Suppl.No.118 (1995), 61-142.

[14] M. Lewin, On a coefficient problem for bi-univalent functions, Appl.Math.Lett.24 (2011), 1569-1573.

[15] A. Mohammed and M. Darus, A generalized operator involving the q-hypergeometric function, Mat. Vesnik 65(2013), no. 4, 454â€“465.

[16] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $z < 1$. Arch. Rational Mech. Anal. 32 1969, 100-112.

[17] V. Paatero, Uber die konforme Abbildung von Gebieten deren Rander von beschr Axonkter Drehung sind, Ann. Acad. Sci. Fenn. Ser, A33 No. 9 (1931).

[18] K. Padmanabhan and R. Parvatham, Properties of a class of functions with bounded boundary rotation, Ann. Polon. Math., 31(1975), 311-323.

[19] Slater, L. J., Generalized hypergeometric functions, Cambridge Univ. Press, Cambridge, 1966.

[20] H. M. Srivastava, D. Bansal, Coefficient estimates for a subclass of analytic and bi-univalent functions. J. Egypt. Math. Soc.(2014), 1-4.

[21] H. M. Srivastava, S. Bulut, M. Caglar and N. Yagmur, Coefficient estimates for general subclass of

ISSN2394-3777 (Print)
ISSN2394-3785 (Online)
Available online atwww.ijartet.com

International Journal of Advanced Research Trends in Engineering and Technology (IJARTET)
Vol. 5, Special Issue 9, March 2018

analytic and bi-univalent functions, Filomat27(5)(2013), 831-842.

[22] T. J. Suffridge, A coefficient problem for a class of univalent functions. Michigan Math. J., 16, 1969, 33-42.

[23] Tan. De Lin, Coefficient estimates for bi-univalent functions. An English summary apperes in Chinese Ann. Math. Ser. B 5 (1984), n0. 4, 741-742.

[24] Q.-H. Xu, Y.-C. Gui and H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Appl. Math. Lett. 25(2012), 990-994.

