ISSN 2394-3777 (Print)
ISSN 2394-3785 (Online)
Available online at www.ijartet.com

International Journal of Advanced Research Trends in Engineering and Technology (IJARTET)
Vol. 5, Issue 4, April 2018

Compatible Mappings of Type (R) in Digital Metric Spaces

Deepak Jain
deepakjain.jain6@gmail.com
Department of Mathematics
Deenbandhu Chhotu Ram University of Science and Technology
Murthal, Sonepat-131039, Haryana (India).

Abstract: In this paper, first we introduce the notion of compatible mappings of type(R) in digital metric spaces analogue to the notion of compatible mappings of type(R) in metric spaces. Secondly, we prove a common fixed point theorem for pairs of compatible mappings of type(R) in digital metric spaces.

Keywords: Fixed point, digital image, compatible mappings of type (R), digital metric space.

2010 Mathematical Subject Classification: 47H10, 54H25, 68U10.

I. Introduction

An image or region may be defined as a two dimensional function $f(x, y)$, where x and y are spatial co-ordinates and f denotes the amplitude function (brightness). The value of f at (x, y) is called intensity or gray level of the image at that point. If the intensity values of f are all finite, discrete quantities, then we call it as a digital image. Digital image is composed of a finite number of elements, each of which has a particular location and value. These elements are called pixels. Pixel is the term used to denote the elements of digital image. The first applications of digital images appeared in the newspaper industry. To create a digital image we need to convert the continuous reused data into digital form. This can be done through sampling and quantization.

The elements of 2D image are called pixels. A pixel p at a point(x, y) has four horizontal and vertical neighbours whose coordinates are given by $(x+1, y), (x-1, y), (x, y+1), (x, y-1)$. The set of pixels is called 4 neighbours of p, is denoted by $N_4(p)$ and each pixel is at unit distance from the point(x, y).

The four diagonal neighbours of p have coordinates
$(x+1, y+1), (x-1, y-1), (x-1, y+1), (x+1, y-1)$
, denoted by $N_D(p)$. These points together with the four neighbours are called 8-neighbor of p and denoted by $N_8(p)$.

Let $\mathbb{Z}^n$, $n \in \mathbb{N}$, be the set of points in the Euclidean n dimensional space with integer coordinates.

Definition 1.1 [2] Let l, n be positive integers with $1 \le l \le$ n. Consider two distinct points

$$p = (p_1, p_2, \dots, p_n),\ q = (q_1, q_2, \dots, q_n) \in \mathbb{Z}^n$$

The points p and q are k_l-adjacent if there are at most l indices i such that $|p_i - q_i| = 1$ and for all other indices j, $|p_j - q_j| \neq 1$, $p_j = q_j$.

(i)Two points p and q in $\mathbb{Z}$ are 2-adjacent if |p – q|=1 (see Figure 1).

Figure 1. 2-adjacency

(ii)Two points p and q in $\mathbb{Z}^2$ are

(a) 8-adjacent if the points are distinct and differ by at most 1 in each coordinate i.e., the 4-neighbours of (x, y) are its four horizontal and vertical neighbours $(x \pm 1, y)$ and$(x, y \pm 1)$.

(b) 4-adjacent if the points are 8-adjacent and differ in exactly one coordinate i.e., the 8-neighbours of (x, y) consist of its 4-neighbours together with its four diagonal neighbours $(x+1, y \pm 1)$ and$(x-1, y \pm 1)$.

(iii)Two points p and q in $\mathbb{Z}^3$ are 26-adjacent if the points are distinct and differ by at most 1 in each coordinate. i.e.,

(a) Six faces neighbours $(x \pm 1, y, z)$, $(x, y \pm 1, z)$ and $(x, y, z \pm 1)$.

(b) Twelve edges neighbours $(x \pm 1, y \pm 1, z), (x, y \pm 1, z \pm 1)$.

(c) Eight corners neighbours $(x \pm 1, y \pm 1, z \pm 1)$

(iv)Two points p and q in $\mathbb{Z}^3$ are 18-adjacent if the points are 26-adjacent and differ by at most 2 coordinate. i.e.,

(a) Twelve edges neighbours $(x \pm 1, y \pm 1, z), (x, y \pm 1, z \pm 1)$.

(b) Eight corners neighbours $(x \pm 1, y \pm 1, z \pm 1)$

(v)Two points p and q in $\mathbb{Z}^3$ are 6-adjacent if the points are 18-adjacent and differ in exactly one coordinate. i.e.,

(a) Six faces neighbours $(x \pm 1, y, z)$, $(x, y \pm 1, z)$ and $(x, y, z \pm 1)$

For more details one can refer to [7].

Now we start with digital metric space(X, d, k) where d is usual Euclidean metric on $\mathbb{Z}^n$ and k denote the adjacency relation among the points in $\mathbb{Z}^n$. We notice that the digital plane $\mathbb{Z}^2$ is the set of all points in the plane $\mathbb{R}^2$ having integer coordinates.

Definition 1.2 Let (X, k) be a digital images set. Let d be a function from $(X, k) \times (X, k) \to \mathbb{Z}^n$ satisfying the following:

(i) $d(p, q) \geq 0$,

(ii) $d(p, q) = 0$ iff $p = q$,

(iii) $d(p, q) = d(q, p)$,

(iv) $d(p, z) = d(p, q) + d(q, z)$,where $p, q, z \in (X, k)$.

A point p of digital image is called pixel and this point has co-ordinates of type (x, y).

The various Euclidean distances between pixels p and q are defined as

(a)$d_2(p, q) = \sqrt[2]{[(x - s)^2 + (y - t)^2]}$

(b)$d_4(p, q) = |x - s| + |y - t|$(City block distance)

(c) $d_8(p, q) = max.(\ |x - s|,\ \ |y - t|)$(Chessboard distance).

II. Topology Of Digital Metric Spaces

In 1999 Boxer [2] defined a k – neighbour of a point $p \in \mathbb{Z}^n$ and gave the digital version of topologies.

A k – neighbour of a point $p \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is k - adjacent to p, where $k \in \{2,4,6\ 8,18,26\}$ and $n \in \{1, 2, 3\}$.

The set $N_k\ (p) = \{q \mid q$ is k - adjacent to $p\}$ is called the k -neighbourhood of p.

In 1994 Boxer [1] defined a digital interval as

$$[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \leq z \leq b\},$$

where a, b $\in \mathbb{Z}$ and a < b.

A digital image $X \subset \mathbb{Z}^n$ is k-connected if and only if for every pair of distinct points $x, y \in X$, there is a set $\{x_0, x_1, x_2, \ldots, x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$ where x_i and x_{i+1} are k -neighbours and i = 0, 1... r- 1, see [5].

The notion of digital continuity in digital topology was developed by Rosenfeld [10] to study 2D and 3D digital images. Further, Ege and Karaca [3] described the digital continuous functions.

Definition 2.1 [1] Let $(X, k_0) \subset \mathbb{Z}^{n_0}$, $(Y, k_1) \subset \mathbb{Z}^{n_1}$ be digital images and f: $X \to Y$ be a function.

(i) If for every k_0-connected subset U of X, $f\ (U)$ is a k_1-connected subset of Y, then f is said to be (k_0, k_1)-continuous.

(ii) f is (k_0, k_1)-continuous for every k_0-adjacent points $\{x_0, x_1\}$ of X, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are k_1-adjacent in Y .

(iii) If f is (k_0, k_1)-continuous, bijective and f^{-1} is (k_0, k_1)-continuous, then f is called (k_0, k_1)-isomorphism and denoted by $\cong_{(k_0, k_1)} Y$.

Proposition 2.2 [4] Let (X, d, k) be a digital metric space. A sequence $\{x_n\}$ of points of a digital metric space (X, d, k) is

(i) a Cauchy sequence if and only if there is $\alpha \in \mathbb{N}$ such that for all, n,m $\gneqq \alpha$, then

$$d(x_n, x_m) \lneqq 1$$

i.e., $x_n = x_m$.

(ii) convergent to a point $l \in X$ if for all $\epsilon \gneqq 0$, there is $\alpha \in \mathbb{N}$ such that for all $n \gneqq \alpha$ then $d(x_n, l) \lneqq \epsilon, i.e. x_n = l$.

Proposition 2.3 [4] A sequence $\{x_n\}$ of points of a digital metric space (X, d, k) converges to a limit $l \in X$ if there is $\alpha \in \mathbb{N}$ such that for all $n \gneqq \alpha$, then $x_n = l$.

ISSN 2394-3777 (Print)
ISSN 2394-3785 (Online)
Available online at www.ijartet.com

International Journal of Advanced Research Trends in Engineering and Technology (IJARTET)
Vol. 5, Issue 4, April 2018

Theorem 2.4 [4] A digital metric space (X, d, k) is always complete.

Definition 2.5 [3] Let (X, d, k) be any digital metric space. A self map f on a digital metric space is said to be digital contraction, if there exists a λ∈ [0, 1) such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq \lambda\, d(x, y)$$

Proposition 2.6 [3] Every digital contraction map $f: (X, d, k) \rightarrow (X, d, k)$ is digitally continuous.

Proposition 2.7 [4] Let (X, d, k) be a digital metric space. Consider a sequence $\{x_n\} \subset X$ such that the points in $\{x_n\}$ are k adjacent. The usual distance $d(x_i, x_j)$ which is greater than or equal to 1 and at most √ t depending on the position of the two points where t ∈ Z^+.

III. Preliminaries

It was the turning point in the fixed point theory literature when the notion of commutativity mappings was used by Jungck [6] to obtain a generalization of Banach's fixed point theorem for a pair of mappings. This result was further generalized, extended and unified using various types of contractions and minimal commutative mappings. In 1995, Pathak et. al. [8] introduced the notion of compatible mappings of type (P). In 2004, Rohan et al. [9] introduced the concept of compatible mappings of type (R) by combining the definition of compatible mappings and compatible mappings of type (P).

Now we give some basic definitions and results that are useful for proving our main results.

Definition 3.1 Let $\emptyset \neq X \subset \mathbb{Z}^n$, $n \in N$ and (X, k) be a digital image and k is an adjacency relation in X. Two self mappings f and g of a digital metric space (X, d, k) are called digitally compatible of type (R) if

$$lim_n d(fgx_n, gfx_n) = 0 \text{ and } lim_n d(ffx_n, ggx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $lim_n fx_n = lim_n gx_n = t$ for some t in X.

Proposition 3.2 Let f and g be digitally compatible mappings of type (R) of a digital metric space (X, d, k) into itself. If $ft = gt$ for some $t \in X$, then $fgt = fft = ggt = gft$.

Proposition 3.3 Let f and g be digitally compatible mappings of type (R) of a digital metric space (X, d, k) into itself. Suppose that $lim_n fx_n = lim_n gx_n = t$ for some t ∈ X. Then

(a) $lim_n gfx_n = ft$ if f is digitally continuous at t.

(b) $lim_n fgx_n = gt$ if g is digitally continuous at t.

(c) $fgt = gft$ and $ft = gt$ if f and g both are digitally continuous at t.

IV. Main Result

Now we prove a common fixed point theorem for pairs of compatible mappings of type (R) in digital metric spaces as follows:

Theorem 4.1 Let $\emptyset \neq X \subset \mathbb{Z}^n$, $n \in N$ and (X, k) be a digital image and k is an adjacency relation in X. Let A, B, S and T be mappings of a digital metric space (X, d, k) into itself satisfying the following conditions:

(C1) $S(X) \subset B(X), T(X) \subset A(X)$,

(C2)
$$d(Sx, Ty) \leq \alpha max\{d(Ax, By), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ax, Ty)\}$$
for all $x, y \in X$, where $\alpha \in (0, 1)$,

(C3) one of the mappings A, B, S and T is continuous.

Assume that the pairs (A, S) and (B, T) are compatible of type R. Then A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be any arbitrary point. From (C1) we can find x_1 such that $S(x_0) = B(x_1) = y_0$ for this x_1 one can find $x_2 \in X$ such that $T(x_1) = A(x_2) = y_1$.Continuing in this way, one can construct a sequence $\{y_n\}$ such that

$$y_{2n} = S(x_{2n}) = B(x_{2n+1}), y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2})$$

for each $n \geq 0$.

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(S(x_{2n}), T(x_{2n+1})) \\ &\leq \alpha max\{d(A(x_{2n}), B(x_{2n+1})), d(A(x_{2n}), S(x_{2n})), d(B(x_{2n+1}), T(x_{2n+1})), \\ &\quad d(S(x_{2n}), B(x_{2n+1})), d(A(x_{2n}), T(x_{2n+1}))\} \\ &\leq \alpha max\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n}), d(y_{2n-1}, y_{2n+1})\} \\ &\leq \alpha max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\}. \end{aligned}$$

On putting $d_n = d(y_n, y_{n+1})$, we have

$$d_{2n} \leq \alpha max\{d_{2n-1}, d_{2n}, d_{2n-1} + d_{2n}\}.$$

Let $d_{2n} > d_{2n-1}$.

Therefore, $d_{2n} \leq 2\alpha d_{2n}$, for all $\alpha \in (0, 1)$, which is a contradiction.

Hence, $d_{2n} \leq d_{2n-1}$.

Let $m, n \in N$ such that $m > n$, we get

$$d(y_m, y_n) \leq \alpha d(y_m, y_{m-1}) + \ldots + \alpha d(y_{n+1}, y_n)$$

ISSN 2394-3777 (Print)
ISSN 2394-3785 (Online)
Available online at www.ijartet.com

International Journal of Advanced Research Trends in Engineering and Technology (IJARTET)
Vol. 5, Issue 4, April 2018

$\leq \alpha^n(d(y_1, y_0)) \to 0$ as $m, n \to \infty$.

This implies that $\{y_n\}$ is a digitally Cauchy sequence in digital metric space (X, d, k). Therefore, $\{y_n\}$ converges to a point z as $n \to \infty$. Consequently the subsequence's $\{Sx_{2n}\}, \{Ax_{2n}\}, \{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converges to the same point z.

Now suppose that A is continuous. Since the mappings A and S are digitally compatible of type (R) it follows from the Proposition 3.3, that $\{AAx_{2n}\}$and $\{SAx_{2n}\}$ converges to Az as $n \to \infty$.

Now we claim that $z = Az$. For this put $x = Ax_{2n}$ and $y = x_{2n+1}$ in (C2) we get

$$d(SAx_{2n}, Tx_{2n+1}) \leq \alpha\max\{d(AAx_{2n}, Bx_{2n+1}), d(AAx_{2n}, SAx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), d(SAx_{2n}, Bx_{2n+1}), d(AAx_{2n}, Tx_{2n+1})\}.$$

Letting n→∞, we have

$$d(Az, z) \leq \alpha\max\{d(Az, z), d(Az, Az), d(z, z), d(Az, z), d(Az, z)\}$$
$$\leq \alpha d(Az, z),$$

implies that $Az = z$.

Next we claim that $Sz = z$.

Putting $= z$, $y = x_{2n+1}$ in (C2) we have

Consider

$$d(Sz, Tx_{2n+1}) \leq \alpha\max\{d(Az, Bx_{2n+1}), d(Ax, Sx_{2n+1}), d(Bx_{2n+1}, Tx_{2n+1}), d(Sz, Bx_{2n+1}), d(Az, Tx_{2n+1})\}.$$

Letting n→∞, we have

$$d(Sz, z) \leq \alpha\max\{d(z, z), d(z, Sz), d(z, z), d(Sz, z), d(Sz, z)\}\}$$
$$\leq \alpha d(Sz, z)$$

implies that $Sz = z$.

Since $SX \subset BX$ and hence there exists a point u in X such that $z = Sz = Bu$.

Now we claim that $z = Tu$.

$$d(z, Tu) = d(Sz, Tu) \leq \alpha\max\{d(Az, Bu), d(Az, Sz), d(Bu, Tu), d(Sz, Bu), d(Az, Tu)\}$$
$$\leq \alpha\max\left\{\begin{matrix} d(z, z), d(z, z), d(z, Tu), \\ d(z, z), d(z, Tu) \end{matrix}\right\}.$$

This implies that $z = Tu$.

Since (B, T) is compatible of type (R) and $Bu = Tu = z$, so by Proposition 3.2, we have $d(BTu, TBu) = 0$ and hence $Bz = BTu = TBu = Tz$.

Also, from (C2), we have

$$d(z, Bz) = d(Sz, Tz) \leq \alpha\max\{d(Az, Bz), d(Az, Sz), d(Bz, Tz), d(Sz, Bz), d(Az, Tz)\}$$
$$\leq \alpha\max\{d(z, Bz), d(z, z), d(Bz, Bz), d(z, Bz), d(z, Bz)\}.$$

This implies that $z = Bz$.

Hence, $z = Bz = Tz = Az = Sz$.

Therefore, z is common fixed point of A, B, S and T.

Similarly, we can also complete the proof by taking B is continuous.

Next suppose that S is continuous.

Since A and S are compatible of type (R) and by Proposition 3.3, we have SSx_{2n} and ASx_{2n} converges to Sz as $n \to \infty$.

We claim that $z = Sz$.

Putting $x = Sx_{2n}$, $y = x_{2n+1}$ in inequality (C2) we have

$$d(SSx_{2n}, Tx_{2n+1}) \leq \alpha\max\{d(ASx_{2n}, Bx_{2n+1}), d(ASx_{2n}, SSx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}), d(SSx_{2n}, Bx_{2n+1}), d(ASx_{2n}, Tx_{2n+1})\}$$

Letting n→∞, we get

$$d(Sz, z) \leq \alpha\max\{d(Sz, z), d(Sz, Sz), d(z, z), d(Sz, z), d(Sz, z)\}$$
$$= \alpha d(Sz, z),$$

implies $Sz = z$.

Since $SX \subset BX$ and hence there exists a point w in X such that $z = Sz = Bw$.

We claim that $z = Tw$.

On putting $x = Sx_{2n}$, $y = w$ in (C2) we have

$$d(SSx_{2n}, Tw) \leq \alpha\max\{d(ASx_{2n}, Bw), d(ASx_{2n}, SSx_{2n}), d(Bw, Tw), d(SSx_{2n}, Bw), d(ASx_{2n}, Tw)\}$$

i.e,
$$d(z, Tw) \leq \alpha\max\{d(z, z), d(z, z), d(z, Tw), d(z, z), d(z, Tw)\}$$
$$= \alpha d(z, Tw),$$

implies that $z = Tw$.

Since B and T are compatible of type (R) on X and $Bw = Tw = z$, so by Preposition 3.2, we have

$d(BTw, TBw) = 0$ and hence $Bz = BTw = TBw = Tz$.

Next, we claim that $z = Tz$.

Putting x = x_{2n} , y = z in (C2) ,we have

$$d(Sx_{2n}, Tz) \leq \alpha max\{ d(Ax_{2n}, Bz), d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), d(Sx_{2n}, Bz), d(Ax_{2n}, Tz)\}$$

i.e, $d(z, Tz) \leq \alpha max\{ d(z, Tz), d(z, z), d(Tz, Tz), d(z, Tz), d(z, Tz)\}$

$= \alpha d(z, Tz)$,

implies that $Tz = z$.

Since $TX \subset AX$, so there exists a point p in X such that $z = Tz = Ap$.

We claim that z = Sp.

Putting $x = p, y = z$ in (C2) we have

$$d(Sp, z) = d(Sp, Tz) \leq \alpha max\{ d(Ap, Bz), d(Ap, Sp), d(Bz, Tz), d(Sp, Bz), d(Ap, Tz)\}$$

$$\leq \alpha max\{ d(z, z), d(z, Sp), d(Tz, Tz), d(Sp, z), d(z, z)\},$$

implies that $Sp = z$.

Since A and S are compatible of type (R) on X, $Sp = Ap = z$, so by Proposition 3.2, we have $d(ASp, SAp) = 0$ and hence $Az = ASp = SAp = Sz$.

That is, $z = Az = Sz = Bz = Tz$.

Therefore, z is common fixed point of A, S, B and T.

Similarly, we can complete the proof when T is continuous.

Uniqueness follows easily.

ACKNOWLEDGEMENT

The author is grateful to the Council of Scientific and Industrial Research, New Delhi for providing fellowship vide file No. 09/1063/0009/2015-EMR-1.

REFERENCES

[1]. L. Boxer, A classical construction for the digital fundamental group, *J. Math. Imaging Vis.*, 10 (1999), 51-62.

[2]. L. Boxer, Digital products, wedges and covering spaces, *J.* Math. Imaging Vis., 25(2006), 159-171.

[3]. O. Ege and I. Karaca, Banach fixed point theorem for digital images, J. Nonlinear Sci. Appl., 8 (2015), 237-245.

[4]. S.E. Han, Banach fixed point theorem from the viewpoint of digital topology, J. Nonlinear Sci. Appl., 9 (2015), 895-905.

[5]. G.T. Herman, Oriented surfaces in digital spaces, CVGIP: Graphical Models and Image Processing, 55 (1993), 381-396.

[6]. G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly **83** (1976), 261–263.

[7]. C. Park, O. Ege, D. Jain, S. Kumar and D. Y. Shin, Commuting and Compatible Mappings in Digital metric spaces, Journal of computational analysis and applications. (Appeared).

[8]. H.K. Pathak, Y.J. Cho, S.M. Kang, B.S. Lee, Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming, Matematiche, 50 (1995), 15-33.

[9]. Y. Rohan, M. R. Singh and L. Shambhu, Common fixed points of compatible mapping of type (R) in Banach spaces, Proc. Math. Soc., Banaras Hindu University 20 (2004), 77-87.

[10]. A. Rosenfeld, Continuous functions on digital pictures, Pattern Recognition Letters, 4 (1986), 177- 184.