



Numerical solutions of Parabolic Partial differential Equation

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ABSTRACT

In this presentation we discussed the classification of partial differential equations and Numerical solutions of PARABOLIC PARTIAL DIFFERENTIAL EQUATION in detailed. We presented numerical solutions of parabolic partial differential equations and discussed only Explicit method one example. A PDE is changed to a system of difference equations that can be solved by means of iterative techniques (Finite Deference Methods). These methods can be divided into two main groups, namely: explicit and implicit methods. There are also other methods that attempt to approximate solutions by polynomial functions[1]

1. Introduction

In this presentation we discussed the classification of partial differential equations and Numerical solutions of PARABOLIC PARTIAL DIFFERENTIAL EQUATION in detailed. We presented numerical solutions of parabolic partial differential equations and discussed only Explicit method one example. A PDE is changed to a system of difference equations that can be solved by means of iterative techniques (Finite Deference Methods). These methods can be divided into two main groups, namely: explicit and implicit methods. There are also other methods that attempt to approximate solutions by polynomial functions

The three most widely used numerical methods to solve PDEs are the finite element method (FEM), finite volume methods (FVM) and finite difference methods (FDM). The FEM has a prominent position among these methods and especially its exceptionally efficient higher-order version hp-FEM. Other versions of FEM include the generalized finite element method (GFEM), extended finite element method (EFEM), spectral finite element method (SFEM), mesh free finite element method, discontinuous Gale kin finite element method (DGFEM), etc.[1]

1.1 PARTIAL DIFFERENTIAL EQUATIONS

PDEs are used to formulate problems involving functions of several variables, and are either solved by hand, or used to create a relevant computer model. PDEs can be used to describe a wide variety of phenomena such as sound, heat, electrostatics, electrodynamics, fluid flow, or elasticity[2]

Definition:

A **partial differential equation** (or briefly a **PDE**) is a mathematical equation that involves two or more independent variables, an unknown function (dependent on those variables), and partial derivatives of the unknown function with respect to the independent variables. The *order of a partial differential equation* is the order of the highest derivative involved. A *solution* (or a *particular solution*) to a partial differential equation is a function that solves the equation. Examples; $u_t = \frac{\partial u}{\partial t}$

1.2 classification of second order partial differential equations:

In the second order PDE includes two independent variables. In general second order linear partial differential equations is of the form

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D = 0$$

Where A, B, C, and D are given functions of x and y or constants.

Equations of this type can be classified in one of three ways depending on the value of $B^2 - 4AC$

Case 1: $B^2 - 4AC > 0$

This resultant is called **hyperbolic Partial differential equation**



Case 2: $B^2 - 4AC = 0$

This resultant is called **parabolic Partial differential equation**

Case 3: $B^2 - 4AC < 0$ this resultant is called **Elliptic Partial differential equation**

2. Numerical Analysis:

Numerical analysis is the study of algorithms that use numerical approximations (as opposed to general symbolic manipulations) for the problems of mathematical analysis

Numerical analysis naturally finds applications in all fields of engineering and the physical sciences, but in the 21st century, the life sciences and even arts have adopted elements of scientific computation.

Before the advent of modern computers numerical methods often depended on hand interpolation in large printed tables. Since the mid-20th century, computers calculate the required functions instead. The interpolation algorithm nevertheless may be used as part of the software's for the solving differential equations.

Example:

From a set of tabulated data derived from an experiment, inferences may have to be drawn; or a system of linear algebraic equations is to be solved. That is numerical analysis is to provide efficient methods for obtaining numerical answers

2.1 Numerical partial differential equation: Numerical partial differential equation is a branch of numerical analysis that studies the numerical solution of partial differential equations.

2.2 NUMERICAL METHODS TO SOLVE PDEs:

The three most widely used numerical methods to solve PDEs are the finite element method (FEM), finite volume methods (FVM) and finite difference methods (FDM). The FEM has a prominent position among these methods and especially its exceptionally efficient higher-order version hp-FEM. Other versions of FEM include the generalized finite element method (GFEM), extended finite element method (XFEM), spectral finite element method (SFEM), mesh free finite element method, discontinuous Galerkin finite element method (DGFEM), etc.

Example:

- Finite Element Method
- Finite Difference Method
- Finite Volume Method

2.3 NATURE OF NUMERICAL SOLUTIONS:

From the points of view of mathematicians, the ideal form of the solution to an initial value problem would be a formula is known, it is usually relatively easy to produce any other form of the solution you may desire, such as a graphical solution or a numerical solution in the form of a table of values. A formulaic solution is not always easy, and in many cases is absolutely impossible.

Several numerical methods have been proposed for the solution of PDE, But only the Finite difference methods have become popular. The approximation of derivatives by finite differences plays a central role in finite difference for the numerical solution of differential equations. [3] proposed a principle in which another NN yield input control law was created for an under incited quad rotor UAV which uses the regular limitations of the under incited framework to create virtual control contributions to ensure the UAV tracks a craved direction. Utilizing the versatile back venturing method, every one of the six DOF are effectively followed utilizing just four control inputs while within the sight of un demonstrated flow and limited unsettling influences. Elements and speed vectors were thought to be inaccessible, along these lines a NN eyewitness was intended to recoup the limitless states. At that point, a novel NN virtual control structure which permitted the craved translational speeds to be controlled utilizing the pitch and the move of the UAV. At long last, a NN was used in the figuring of the real control inputs for the UAV dynamic framework. Utilizing Lyapunov systems, it was demonstrated that the estimation blunders of each NN, the spectator, Virtual controller, and the position, introduction, and speed following mistakes were all SGUUB while unwinding the partition Principle.

2.3.1 Finite difference approximations

- A finite difference method typically involves the following steps:
- Generate a grid, for example $(x_i, t^{(k)})$, where we want to find an approximate solution.
- Substitute the derivatives in an ODE/PDE or an ODE/PDE system of equations with finite deference schemes. The ODE/PDE then become a linear/non-linear system of algebraic equations.



- Solve the system of algebraic equations.
- Implement and debug the computer code.
- Do the error analysis, both analytically and numerically.

Note that besides *Finite Difference* methods, there are other methods that can be used to solve ODE/PDEs such as *Finite element methods*, *spectral methods* etc. Generally finite difference are simple to use for problems defined on regular geometries, such as an interval in 1D, a rectangular domain in two space dimensions, and a cubic in three space dimensions.

Finite difference methods for heat equation and related PDEs:

Second order (i)Explicit and (ii)Implicit methods

3. Numerical solutions of parabolic PDE

3.1 Numerically Defining parabolic partial differential equation

The general form for a second order linear PDE with two independent variables and one dependent variable is

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D = 0$$

The criterion for an equation of this type to be considered parabolic is $B^2 - 4AC = 0$

3.2. Physical example of an parabolic partial differential equations



Figure:1

The temperature $T(x,t)$ of a metal rod exposed to two different temperatures at each end can be found using the heat conduction .

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

3.3 Example for parabolic partial differential equations

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D = 0$$

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

$$A = \alpha, B = 0, C = 0$$

$$B^2 - 4AC = 0$$

3.4 Discretizing the parabolic partial differential equations

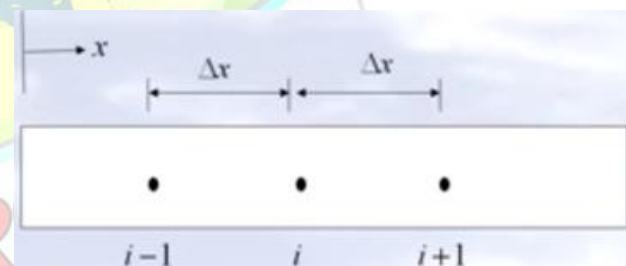


Figure:2

For a rod of length L divided into $n+1$ nodes $\Delta x = L/n$. The time is similarly broken into time steps of Δt . Hence T_i^j corresponds to the temperature at node i , i.e. $x = (i)(\Delta x)$ and $t = (j)(\Delta t)$

3.5 Approximating left hand side

If we define $\Delta x = L/n$ we can then write the finite central divided difference Approximation of the

left hand side at a general interior node (i) as



$$\left. \frac{\partial^2 T}{\partial x^2} \right|_{i,j} \cong \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{(\Delta x)^2}$$

Where (j) is the node number along the time by referring Figure 2

3.6 Approximating right hand side

The time derivative on the right hand side is approximately by the forward divided difference method as

$$\left. \frac{\partial T}{\partial t} \right|_{i,j} \cong \frac{T_i^{j+1} - T_i^j}{\Delta t} \text{ by referring Figure 2}$$

4. Explicit Methods to Solve PDEs

- 1) possible to solve (at a point) directly for all unknown values in the finite difference scheme
- 2) stable only for certain time step sizes (or possibly never stable!). Stability can be checked using Fourier or von Neumann analysis. Timestep size governed by Courant condition for wave equation

4.1 EXPLICIT METHOD-EXAMPLE

Explicit method for parabolic PDE: Theory

The general form for a second order linear PDE with two independent variables and one dependent variable is

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D = 0$$

The criterion for an equation of this type to be considered parabolic is $B^2 - 4AC = 0$

4.1.1 Physical example of an parabolic partial differential equations: The Internal temperature of a metal rod exposed to known temperature at each end can be found using the heat conduction equation by referring figure 1

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

$$T(0, t) = T_l$$

$$T(L, t) = T_r$$

$$T(x, 0) = T_b$$

4.1.2 Discretizing the parabolic partial differential equations:

For a rod of length L divided into n+1 nodes $\Delta x = L/n$. The time is similarly broken into time steps of Δt . Hence T_i^j corresponds to the temperature at node I, i.e $x = (i)(\Delta x)$ and $t = (j)(\Delta t)$ by referring figure 2

4.1.3 Approximating left hand side If we define $\Delta x = L/n$ we can then write the finite central divided difference Approximation of the left hand side at a general interior node (i) as

$$\left. \frac{\partial^2 T}{\partial x^2} \right|_{i,j} \cong \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{(\Delta x)^2} \quad \text{Where (j) is the node number along the time}$$

$$\left. \frac{\partial^2 T}{\partial x^2} \right|_{i,j} \cong \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{(\Delta x)^2}$$

The time derivative on the right hand side is approximately by the forward divided difference method as

$$\left. \frac{\partial T}{\partial t} \right|_{i,j} \cong \frac{T_i^{j+1} - T_i^j}{\Delta t}$$

Substituting these approximation into the governing equation

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

$$\alpha \frac{T_{i+1}^j - 2T_i^j + T_{i-1}^j}{(\Delta x)^2} = \frac{T_i^{j+1} - T_i^j}{\Delta t}$$

Solving for the temperature at the time node j+1 gives

$$T_i^{j+1} = T_i^j + \alpha \frac{\Delta t}{(\Delta x)^2} [T_{i+1}^j - 2T_i^j + T_{i-1}^j]$$

$$\text{Let } \lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$$



$$T_i^{j+1} = T_i^j + \lambda [T_{i+1}^j - 2T_i^j + T_{i-1}^j]$$

Equation can be solved explicitly because it can be written for each internal location node of the rod for time node j+1 in terms of the temperature at time node j. [5]

5 Explicit method for parabolic PDE:

Example: Consider a steel rod that is subjected to a temperature of 100°C on the left end and 25°C on the right end. If the rod is of length 0.05m, use the explicit method to find the temperature distribution in the rod from $t=0$ and $t=9\text{s}$ seconds. Use $\Delta x = 0.01\text{m}$, $\Delta t = 3\text{s}$.

Sol: Given

$$k = \frac{54\text{W}}{\text{m-K}}, \rho = 7800\text{kg/m}^3, c = 490 \frac{\text{J}}{\text{kg-K}}$$

The initial temperature of rod is 20°C

$$T_i^{j+1} = T_i^j + \lambda [T_{i+1}^j - 2T_i^j + T_{i-1}^j]$$

$$\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$$

Equation can be solved explicitly because it can be written for each internal location node of the rod for time node j of the +1 in terms of the temperature at time node

Nodal temperature when $t=0\text{sec}$, $j=0$

$$\left. \begin{aligned} T_0^0 &= 100^\circ\text{C} \\ T_1^0 &= 20^\circ\text{C} \\ T_2^0 &= 20^\circ\text{C} \\ T_3^0 &= 20^\circ\text{C} \\ T_4^0 &= 20^\circ\text{C} \\ T_5^0 &= 25^\circ\text{C} \end{aligned} \right\} \text{interior node}$$

$$\lambda = \alpha \frac{\Delta t}{(\Delta x)^2} = \frac{54}{7800 \times 490} \times \frac{3}{0.01 \times 0.01} = 0.4239$$

We can now evaluate the temperature at each node explicitly using the equation formulated earlier

$$T_i^{j+1} = T_i^j + 0.4239 [T_{i+1}^j - 2T_i^j + T_{i-1}^j]$$

Finding temperature at $i=1, t=3\text{s}$

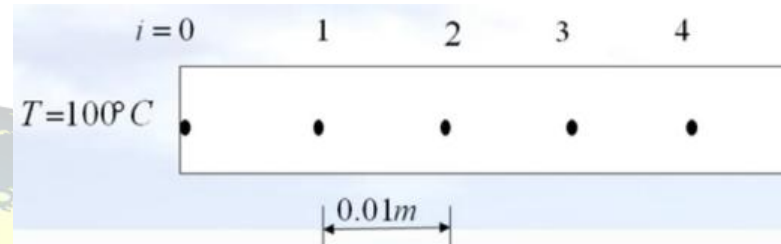


Figure 3

At $t=3\text{s}$ by setting $j=0$

$i=0$ $T_0^1 = 100^\circ\text{C}$ --- Boundary condition

$$i=1 \quad T_1^1 = T_1^0 + 0.4239 [T_2^0 - 2T_1^0 + T_0^0]$$

$$= 20 + 0.4239 (20 - 2(20) + 100)$$

$$= 20 + 0.4239 (80)$$

$$= 53.912^\circ\text{C}$$

$i=2$

$$T_2^1 = T_2^0 + 0.4239 [T_3^0 - 2T_2^0 + T_1^0]$$

$$= 20 + 0.4239 (20 - 2(20) + 20)$$

$$= 20^\circ\text{C}$$

$$T_i^{j+1} = T_i^j + 0.4239 [T_{i+1}^j - 2T_i^j + T_{i-1}^j]$$

At $t=3\text{s}$



$$T_0^1 = 100^\circ C \text{ --- boundary condition}$$

$$\left. \begin{array}{l} T_1^1 = 53.912^\circ C \\ T_2^1 = 20^\circ C \\ T_3^1 = 20^\circ C \\ T_4^0 = 22.120^\circ C \end{array} \right\} \text{interior node}$$

$$T_5^0 = 25^\circ C \text{ --- boundary condition}$$

Finding temperature at $j=2, t=6s$

At $t=6s$ by setting $j=1$

$$i=0 \quad T_2^0 = 100^\circ C$$

$$i=1 \quad T_1^2 = T_1^1 + 0.4239[T_2^1 - 2T_1^1 + T_0^1] = 53.9$$

$$12 + 0.4239(20 - 2(53.912) + 100)$$

$$= 53.912 + 0.4239(12.176)$$

$$= 59.073^\circ C$$

$$T_2^2 = T_2^1 + 0.4239[T_3^1 - 2T_2^1 + T_1^1] = 20 + 0.4239(20 - 2(20) + 53.912)$$

$$= 34.375^\circ C$$

At

$$T_0^1 = 100^\circ C \text{ --- boundary condition}$$

$$\left. \begin{array}{l} T_1^2 = 59.073^\circ C \\ T_2^2 = 34.375^\circ C \\ T_3^2 = 20.889^\circ C \\ T_4^0 = 22.442^\circ C \end{array} \right\} \text{interior node}$$

$$T_5^0 = 25^\circ C \text{ --- boundary condition}$$

Finding temperature at

$j=3, t=9s$

At $t=9s$ by setting $j=2$

$$i=0 \quad T_2^1 = 100^\circ C$$

$$T_1^3 = T_1^2 + 0.4239[T_2^2 - 2T_1^2 + T_0^2]$$

$$= 59.073 + 0.4239(34.375 - 2(59.073) + 100)$$

$$= 65.953^\circ C$$

$i=2$

$$T_2^3 = T_2^2 + 0.4239[T_3^2 - 2T_2^2 + T_1^2] = 34.375 - 0.4239[20.889 - 2(34.375) + 59.073] = 39.132^\circ C$$

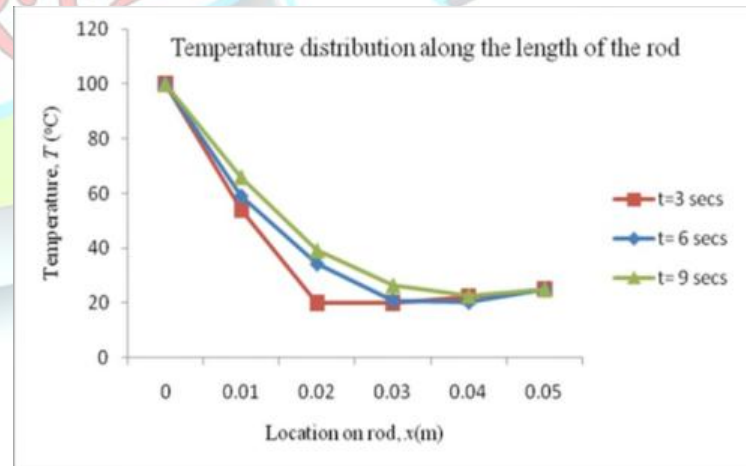
At

$t=3s$

$$T_0^1 = 100^\circ C \text{ --- boundary condition}$$

$$\left. \begin{array}{l} T_1^3 = 65.953^\circ C \\ T_2^3 = 39.132^\circ C \\ T_3^3 = 27.266^\circ C \\ T_4^3 = 22.872^\circ C \end{array} \right\} \text{interior node}$$

$$T_5^1 = 25^\circ C \text{ --- boundary condition}$$



Conclusion:

Numerical solutions for the parabolic partial differential equations are obtained using



Explicit method and also by implicit for finding heat conduction to get different temperatures distribution along the length of the rod and this methods have

REFERENCES

- [1] S.S. SAATRY: Introductory Methods Of numerical Analysis, Jwamer, Karwan HF, and Faraidun K. Hama-Salh. "On Optimality of Lacunary Interpolation for Recovery of C6 Seventh Degree Spline." *International Journal of Open Problems in Computer Science and Mathematics* 5.1 (2012): 103-113.
- [2] M.K. JAIN : Numerical Solutions of differential equations, Mahinder Kumar. "Numerical solution of differential equations." (1979)
- [3] Christo Ananth, "A Novel NN Output Feedback Control Law For Quad Rotor UAV", *International Journal of Advanced Research in Innovative Discoveries in Engineering and Applications* [IJARIDEA], Volume 2, Issue 1, February 2017, pp: 18-26.
- [4] Y. Pinchover & Rubinstein, *An Introduction to Partial Differential Equations*, New York: Cambridge University Press. Rubinstein, J. Hyam, and Robert Sinclair. "Visualizing Ricci flow of manifolds of revolution." *Experimental Mathematics* 14.3 (2005): 285-298.
- [5] Applied Numerical Methods for Engineers using Matlab and C, R. J. Schilling and S. L. Harris - Schilling, Harris A., and Sandra L. Harris. *Applied numerical methods for engineers using MATLAB*. Brooks/Cole Publishing Co., 1999.
- [6] Computational Physics Problem solving with computers, R.H. Landau and M. L. Paez. Paez, M. J. "Computational Physics, Problem Solving With Computers." (1997).
- [7] An Introduction to Computational Physics, T. Pang. Pang, Tao. "An introduction to computational physics." (1999): 94-95.
- [8] Numerical Recipes in Fortran (2nd Ed.), W. H. Press et al. Press, William H., et al. "Numerical recipes in Fortran: diskettes 3.5" (720K) for IBM PC, PS/2 and compatibles (DOS)." (1992)
- [9] Introduction to Partial Differential Equations with Matlab, J. M. Cooper. Cooper, J. M. "Introduction to partial differential equations with matlab birkhauser." Boston, MA (1998).
- [10] Numerical solution of partial differential equations, K. W. Morton and D. F. Mayers. Morton, Keith W., and David Francis Mayers. *Numerical solution of partial differential equations: an introduction*. Cambridge university press, 2005.
- [11] Spectral methods in Matlab, L. N. Trefethen. Trefethen, Lloyd N. *Spectral methods in MATLAB*. Society for Industrial and Applied Mathematics, 2000.