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FINAL COMPLEX MEASURE

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ABSTRACT

Measure theory is the branch of mathematics that studies product measures, finite measures, signed measures and complex measures. In mathematics, specifically measure theory a complex measure generalizes the concept of measure by letting it have complex values. In other words, one allows for sets whose size (length, area and volume) is a complex number. Complex measure is the main technical tool in measure theory. Every finite measure can be treated as a complex measure. In this case the relationship between finite measures and complex measures for containing the topics of Radon Nikodym Theorems and Polar representations.

DEFINITIONS

DEFINITION 1:

Let (**X, S**) be a measurable space and $\boldsymbol{\mu}, \boldsymbol{\gamma}$ be two finite signed measure on (**X, S**).

Consider $\boldsymbol{\eta}$: **S→C** defined by

$\boldsymbol{\eta}(\mathrm{E}) = \boldsymbol{\mu}(E) + i\boldsymbol{\gamma}(E),\ E \in \boldsymbol{S}$

Clearly, $\boldsymbol{\eta}(\phi) = 0$,

Let $\{E_n\}_{n\geq 1}$ be a sequence of Let $\mathbf{E} = \bigcup_{n=1}^{\infty} \mathrm{E_n}$. Then the series $\sum_{n=1}^{\infty}(\boldsymbol{\mu}(\mathrm{E_n}) + i\boldsymbol{\gamma}(\mathrm{E_n}))$ is absolutely convergent, and it converges to $\boldsymbol{\mu}(\mathrm{E}) + i\boldsymbol{\gamma}(\mathrm{E})$.

Hence $\sum_{n=1}^{\infty} \boldsymbol{\eta}(\mathrm{E_n})$ is independent of any rearrangement of the series and we may write $\boldsymbol{\eta}(\mathrm{E}) = \sum_{n=1}^{\infty} \boldsymbol{\eta}(\mathrm{E_n})$.

Thus $\boldsymbol{\eta}$ is a **countable additive complex valued** set function on (**X, S**).

DEFINITION 2:

Let (**X, S**) be a measure space. A set function $\boldsymbol{\gamma}$: **S→C** is called a **complex measure** on (**X, S**).

If the following conditions are satisfied:

i) $\boldsymbol{\mu}(\phi) = 0$

ii) $\boldsymbol{\mu}$ is countably additive in the following sense:

if $\mathrm{E} = \bigcup_{n=1}^{\infty} \mathrm{E_n}$, where $E_n'S$ are pairwise disjoint sets from **S**, then $\sum_{n=1}^{\infty} \boldsymbol{\mu}(\mathrm{E_n})$ is absolutely convergent and converges to $\boldsymbol{\mu}(\mathrm{E})$.

We write this as $\boldsymbol{\mu}(\mathrm{E}) = \sum_{n=1}^{\infty} \boldsymbol{\mu}(\mathrm{E_n})$

EXAMPLE:

Every finite signed measure can be treated as a complex measures. The set function $\boldsymbol{\eta} = \boldsymbol{\mu} + i\boldsymbol{\gamma}$,where $\boldsymbol{\mu}, \boldsymbol{\gamma}$ are finite signed measures is a complex measures. Consider a complex valued integrable functionf on (**X, S**, $\boldsymbol{\mu}$) and define

$\gamma(\mathrm{E}) = \int_{\mathrm{E}} \mathrm{f} d\boldsymbol{\mu}, \mathrm{E} \in \mathbf{S}$

Then γ is a complex measure.

In fact, the equality

$\gamma(\mathrm{E}) + \int_E Re(\mathrm{f}) d\boldsymbol{\mu} + i \int_E Im(\boldsymbol{f}) d\boldsymbol{\mu},\ E \in \boldsymbol{S}$

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If $\mu(E)=0$, for some $E\in S$, then clearly $\gamma(E)=0$ as a complex number.

DEFINITION 3:

Let (X, S, μ) be a measure space and γ is a complex measure on **S.** We say γ is **absolutely Continuous** with respect to μ if $\gamma(E)=0$ for all $E\epsilon S$ for which $\mu(E)=0$. We write this as $\gamma \ll \mu$.

DEFINITION 4:

Let γ be a complex measure on (X, S). A complex valued function g on X is said to be **γ-integrable** if $g\in L_1^r(X,S,\gamma i)$, $\forall 1 \le i \le 4$

we write

$$\int g d\gamma = \int g d\gamma_1 - \int g d\gamma_2 + i\int g d\gamma_3 - i\int g\, d\,\gamma_4$$

DEFINITION 5:

Let γ be a complex measure on (X,S) for $E\in S$

Define,

$|\gamma|\,(E) = \sup\{\sum_{i=1}^{n}|\gamma(Ei)| / \{E_1, \ldots E_n\}$ is a measurable pertition of E$\}$

The set function $|\gamma|$ is called the **total variation** of γ.

THEOREMS

THEOREM 1: [Radon – Nikodym theorem for complex Measure]

Let (X, S, μ) be a σ - finite measure space and let γ be a complex measure on **S** such that $\gamma \ll \mu$. Then there exists a complex value function $f\epsilon L_1(X,S,\mu)$ such that

$\int g d\gamma = \int f g d\mu \,\forall g\epsilon \cap_{i=1}^{4} L_1^r(X,S,\gamma_i)$ where $\gamma_i, 1\le i\le 4$ are finite measures on (X, S)

Proof:

Consider $\gamma_1, \gamma_2, \gamma_3, \gamma_4$.

Then $\gamma_i \ll \mu, \forall i=1, 2, 3, 4$.

By theorem,

$$\gamma_i(E) = \int_E f_i d\mu, \forall\, E\epsilon S$$

Let $f = f_1 - f_2 + if_3 - if_4$

Then f is a complex valued measurable function, $f \in L_1(X,S,\mu)$ and

$$\gamma(E) = \int_E f d\mu, \quad \forall E\epsilon S$$

It is easy to show that,

$$\int g d\gamma_i = \int f_i g d\mu$$

Whenever $g \in L_1{}^r(X,S,\gamma_i)$

Let $g\epsilon \cap_{i=1}^{4} L_1^r(X,S,\gamma_i)$

Then eq (1) holds for such a g and each i=1, 2, 3, 4 we have,

$$\int g d\gamma = \int g d\gamma_{1,} - \int g d\gamma_{2,} + i\int g d\gamma_{3,} - i\int g d\gamma_{4,}$$

$$= \int g(f_1 - f_2 + if_3 - if_4)\, d\mu$$

$\int g d\gamma = \int g f d\mu$.

Hence the theorem is proved.

Theorem 2:

Let γ be a complex measure on (X, S). Then there exist finite measures $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ on (X, S) such that $\gamma_1 \perp \gamma_2, \gamma_3 \perp \gamma_4$ and $E\in S$,

$$\gamma(E) = \gamma_1(E) - \gamma_2(E) + i\gamma_3(E) - i\gamma_4(E)$$

Proof:

For every $E\in S$.

Consider,

$(\mathrm{Re}(\gamma))(E)$ = Real part of $\gamma(E)$ and

$(\mathrm{Im}(\gamma))(E)$ = Imaginary part of $\gamma(E)$

Then $(\mathrm{Re}(\gamma))$ and $(\mathrm{Im}(\gamma))$ are finite signed measures on (X, S).

put $\gamma_1 = (Re(\gamma))^+, \gamma_2 = (Re(\gamma))^-, \gamma_3 = (Im(\gamma))^+$ and $\gamma_4 = (Im(\gamma))^-$

Hence the theorem is proved.

THEOREM: 3

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Let $\boldsymbol{\gamma}$ be a complex measure on **(X, S).**

Let $\boldsymbol{\gamma}_i$,1≤ i ≤ 4 be the finite measures $\boldsymbol{\gamma_1}, \boldsymbol{\gamma_2}, \boldsymbol{\gamma_3}, \boldsymbol{\gamma_4}$ on **(X,S).** Then the following hold:-

i) $|\boldsymbol{\gamma}|(E) \leq \sum_{i=1}^{4} \boldsymbol{\gamma}_i(E)$, ∀ E ∈**S**

ii) $|\boldsymbol{\gamma}|$ is a finite measure on **(X,S)**

iii) $|\boldsymbol{\gamma}(E)| \leq |\boldsymbol{\gamma}|(\mathbf{X})$, ∀ E∈**S**

iv) $\boldsymbol{\gamma_i}(E) \leq |\boldsymbol{\gamma}|(E)$, ∀ E ∈**S**

v) L1 $(\mathbf{X},\mathbf{S},|\boldsymbol{\gamma}|) = \bigcap_{i=1}^{4} L_1(\mathbf{X},\mathbf{S},\boldsymbol{\gamma_i})$

vi) For any measure μ on **(X,S)**, $\boldsymbol{\gamma} \ll \boldsymbol{\mu}$ iff $\boldsymbol{\gamma_k} \ll \mu$ iff $|\boldsymbol{\gamma}| \ll \boldsymbol{\mu}$.

Proof:

To prove (i):

It is easily proved

To prove (ii):

First note that, $|\boldsymbol{\gamma}|(\emptyset) = 0$

To prove the countable additivity of $|\boldsymbol{\gamma}|$

Let A= $\bigcup_{j=1}^{\infty} A_j$, where A_j ∈**S ,** ∀ j

and $A_i \cap A_j = \emptyset$ for i≠j.

Let $\propto < |\boldsymbol{\gamma}|(A)$, where ∝∈**R** is arbitrary.

Let $E_1, E_2 \ldots, E_n$ ∈**S** be such that $E_i \cap E_j = \phi$ for i≠j

A=$\bigcup_{j=1}^{n}$ Ej and $\propto < \sum_{j=1}^{n} |\boldsymbol{\gamma}(E_j)|$

Then

$\propto < \sum_{j=1}^{n} \sum_{k=1}^{\infty} |\gamma(E_j \cap A_k)| =$

$\sum_{k=1}^{\infty} \sum_{j=1}^{n} |\boldsymbol{\gamma}(E_j \cap A_k/ \leq \sum_{k=1}^{\infty} |\boldsymbol{\gamma}|(A_k)$

Since $\propto < |\boldsymbol{\gamma}|(A)$ is arbitrary

We have $|\boldsymbol{\gamma}|(A) \leq \sum_{k=1}^{\infty} |\boldsymbol{\gamma}|(A_k)$.

To prove the reverse inequality,

We may assume without loss of generality that, $|\boldsymbol{\gamma}|(A) < +\propto$

It is easy to see that $|\boldsymbol{\gamma}|(E) \leq |\boldsymbol{\gamma}|(F)$

Whenever E⊆F.

Thus $|\boldsymbol{\gamma}|$ (Aj) <+∞, ∀ j

Let ε>0 be given.

Choose ∀j, sets E_k^j ∈**S,** 1≤k≤ k_j

Such that $E_k^j \cap E_l^j = \Phi$ for $k \neq l$

$A_j = \bigcup_{k=1}^{k} E_K^j$ and

$|\boldsymbol{\gamma}|(Aj) - \epsilon/2^j \leq \sum_{j=1}^{k_j} |\boldsymbol{\gamma}(E_k^j)|$

Thus $\forall m$

$\sum_{j=1}^{m} |\boldsymbol{\gamma}|(A_j) \leq \sum_{j=1}^{m} \frac{\epsilon}{2j} + \sum_{j=1}^{m} \sum_{k=1}^{k_j} |\boldsymbol{\gamma}(E_k^j)|$

$< \epsilon + \sum_{j=1}^{m} \sum_{k=1}^{k_j} |\boldsymbol{\gamma}(E_n^j)| +$

$|\boldsymbol{\gamma}(\bigcup_{j=m+1}^{\infty} A_j)|$

$\sum_{j=1}^{m} |\boldsymbol{\gamma}|(A_j) < \epsilon + |\boldsymbol{\gamma}|(A)$

Since this holds ∀ m and ∈> 0

We have $\sum_{j=1}^{\infty} |\boldsymbol{\gamma}|(A_j) < |\gamma|(A)$

$|\boldsymbol{\gamma}|$ is a measure.

Since $|\boldsymbol{\gamma}|(x) \leq \sum_{i=1}^{4} \boldsymbol{\gamma}_i(x) < +\infty$.

By (i), (ii) is a finite measure.

To prove (iii):

Let E ϵ ***S*** be fixed.

Then X = E ∪ E^C and hence

$|\boldsymbol{\gamma}(E)| \leq |\boldsymbol{\gamma}(E)| + |\boldsymbol{\gamma}(E^c)| \leq |\boldsymbol{\gamma}|(x)$

$|\boldsymbol{\gamma}(E)| \leq |\boldsymbol{\gamma}|(X)$

To Prove (iv):

Let A, B be a Hahn decomposition of X with respect to Re($\boldsymbol{\gamma}$)

Then $\forall E \; \epsilon \; \boldsymbol{S}$,

$\gamma_1(E) = (Re(\boldsymbol{\gamma}))^+(E)$

$= (\mathrm{Re}\,(\boldsymbol{\gamma}))(E \cap A_1)$

$= |(\mathrm{Re}\,(\boldsymbol{\gamma}))(E \cap A_1)|$

(1) $\leq |\boldsymbol{\gamma}(E \cap A_1)|$

$\leq |\boldsymbol{\gamma}(E \cap A_1) + |\boldsymbol{\gamma}(E \cap B_1)|$

$\boldsymbol{\gamma}_1(E) \leq |\boldsymbol{\gamma}|(E)$

Similarly,

$\boldsymbol{\gamma}_2(E) = \left(\mathrm{Re}(\boldsymbol{\gamma})\right)^-(E)$

$= |(\mathrm{Re}(\boldsymbol{\gamma})(E \cap B_1)|$

$\leq |\boldsymbol{\gamma}(E \cap B_1)| + |\boldsymbol{\gamma}(E \cap A_1)|$

$\boldsymbol{\gamma}_2(E) \leq |\boldsymbol{\gamma}|(E)$.

That $\boldsymbol{\gamma_3}(E)$ and $\gamma_4(E)$ are both less than $|\boldsymbol{\gamma}|(E)$.

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$\boldsymbol{\gamma_1}(E) \le |\boldsymbol{\gamma}|(E), \ \forall E \ \in \boldsymbol{S}.$

To Prove (v):

Let us first consider f, a non-negative simple measurable function on **(X,S).**

Let $f = \sum_{j=1}^{m} \boldsymbol{\alpha_j} \chi_{E_j}$

Then using (i), we have

$\int f d|\boldsymbol{\gamma}| = \sum_{j=1}^{m} \alpha_j |\boldsymbol{\gamma}|(E_j)$

$$\le \sum_{j=1}^{m} \boldsymbol{\alpha_j} \left(\sum_{k=1}^{4} \boldsymbol{\gamma_k}(E_j) \right)$$

$$\int f d|\boldsymbol{\gamma}| \le \sum_{k=1}^{4} (\int d\boldsymbol{\gamma}_k)$$

Also using (iv), ∀k = 1, 2, 3, 4

We have, $\int f d\boldsymbol{\gamma_k} = \sum_{j=1}^{m} \boldsymbol{\alpha_j} \boldsymbol{\gamma_k}(E_j)$

$$\le \sum_{j=1}^{m} \boldsymbol{\alpha_j} |\boldsymbol{\gamma}|(E_j)$$

$\int f d\boldsymbol{\gamma_k} \le \int f d|\boldsymbol{\gamma}|$

Let f be any non-negative measurable function and $\{ s_n\}_{n\ge1}$ be a sequence of non-negative simple measurable functions on X increasing to $|f| a.e\ |\gamma|$. f replace by s_n gives

$\int f d|\gamma| \le \sum_{k=1}^{n} (\int |f| d\gamma_K)$and

$$\int |f| d\gamma_K \le \int |f| d|\gamma|.$$

Thus $f \epsilon$L1(**X,S**,|γ|) iff$f\epsilon \cap_{k=1}^{4} L_1 (\boldsymbol{X}, \boldsymbol{S}, \boldsymbol{\gamma_k})$

Therefore, L_1 (**X, S,** |γ|) = $\cap_{i=1}^{4} L_1 (\boldsymbol{X}, \boldsymbol{S}, \boldsymbol{\gamma_i})$

To Prove (vi):

Finally if **μ** is a measure and **μ**<<γ, then **μ** (E) = 0,

implies γ(E) = 0.

In Particular, for E**ϵS** fixed with **μ**(E) = 0,

We have **μ** (F) = 0, for every set F⊆ E, F ϵS,

Thus γ (F) = 0, $\forall F \subseteq$ E and

It follows that, $|\gamma|$(E)=0.

Hence $\gamma \ll \mu$ implies $|\gamma| \ll \mu$

Also it follows (iv) that, $\gamma_k \ll |\gamma|, \forall k = 1,2,3,4$

Thus $\gamma_k \ll \mu, \forall k$

Finally if ∀ k, γ k << μ

It follows that, $\gamma \ll \mu$

Hence the theorem is proved.

Theorem 4:

Let (**X, S, μ**) be a σ - finite space and let $f\epsilon$ L1 (**X, S, μ**) be a complex valued function. Let **γ** (E) $=\int_E f d\mu$, Eϵ**S**. Then γ is a complex measure and

∀ Eϵ**S** |γ| (E) $=\int_E |f| d\mu$.

In Particular, |γ| = |γ| (x) = $\int |f| d\mu$ = ||f||.

Proof:

Let E ϵ**S** be fixed.

Then for any measurable partition {F1, F2,………Fn} of E.

$$\sum_{j=1}^{n} |\boldsymbol{\gamma}(F_j)| = \sum_{j=1}^{n} | \int_{F_j} f d\boldsymbol{\mu}| \le \int_E |f| d\mu$$

Hence,

$$|\boldsymbol{\gamma}|(E) \le \int_E |\boldsymbol{f}| d\mu$$

Let $\{\boldsymbol{s_n}\}_{n\ge1}$ be a sequence of non-negative simple functions

Such that$|s_n| \le 1, \forall n$ and **sn** increase to χ_E.

Define for xϵX

$$g_n(x) = \begin{cases} \boldsymbol{s_n}(x) \left(\frac{\overline{f(x)}}{|f(x)|} \right) & \text{if } f(x) \ne 0 \\ 0 & \text{if } f(x) = 0 \end{cases}$$

where $\overline{f(x)}$ denotes the complex conjugate of f(x).

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Then $\{g_n f\}_{n\geq 1}$ converges to $\chi_E\,|f|$ and $|g_n f| \leq |f| \epsilon L_1\,(\boldsymbol{X},\boldsymbol{S},\boldsymbol{\mu})$

Thus by Lebesgue's dominated convergence theorem

we get,

$\int_E |f| d\boldsymbol{\mu} = \lim_{n\to\infty} \int f \boldsymbol{g}_n d\mu$

Also **sn** being a simple function,

Let sn $=\sum_{j=1}^{m} \alpha_j \chi_{E_j}$,

Where {E1, E2,…..En} is a measurable partition of X.

Since $s_n \leq \boldsymbol{\chi}_E, s_n(x) = 0\ \forall x \in X\backslash E$ and $0 \leq \alpha_j \leq 1, \forall_j$.

Thus

$$|\int f s_n d\boldsymbol{\mu}| = |\int_E f s_n d\boldsymbol{\mu} = \left|\sum_{j=1}^{m} \boldsymbol{\alpha}_j \int_{E\cap Ej} f d\,\boldsymbol{\mu}\right| \leq \sum_{j=1}^{m} |\int_{E\cap Ej} f d\boldsymbol{\mu}|$$

$$=\sum_{j=1}^{m} |\boldsymbol{\mu}(E\cap Ej)|$$

$|\int f s_n d\boldsymbol{\mu}| \leq |\boldsymbol{\mu}|(E)$

From (1), (2) and (3), we get

$$|\boldsymbol{\mu}|(E) = \int_E |f| d\boldsymbol{\mu}$$

In particular, with E = X

$$\left||\boldsymbol{\mu}|\right| = \left||f|\right|.$$

Corollary 5: [polar representation]

Let $\boldsymbol{\mu}$ be a complex measure on **(X, S).** Then there exists a measurable function f such that $|f(x)| = 1$ for all $x \in \boldsymbol{X}, and\ E \in \boldsymbol{S}$,

$$\boldsymbol{\mu}(E)=\int_E f d|\boldsymbol{\mu}|.$$

Proof:

Clearly, $\boldsymbol{\mu} \ll |\boldsymbol{\mu}|$.

Let f be the function,

$\forall$E$\in \boldsymbol{S}$

$\boldsymbol{\mu}(E)=\int_E f d|\boldsymbol{\mu}|$.

Let A = $\{x \in \boldsymbol{X}/|f(x)| < 1\}$ and

B = $\{x \in \boldsymbol{X}/|f(x)| > 1\}$

Then by theorem,

$$\int_A (1-|f|) d|\mu| = |\boldsymbol{\mu}|(A) - \int_A |f| d|\boldsymbol{\mu}| = 0$$

This implies that,

$$|\boldsymbol{\mu}|(A) = 0$$

Similarly, $\int_B (|f|-1) d|\boldsymbol{\mu}| = \int_B |f| d|\boldsymbol{\mu}| - |\boldsymbol{\mu}|(B) = 0$

Implies that, $|\boldsymbol{\mu}|(B) = 0$

Hence $|f(x)| = 1$ for a.e $(|\boldsymbol{\mu}|)\ x \in \boldsymbol{X}.$

$$\boldsymbol{\mu}(E) = \int_E f d|\boldsymbol{\mu}|.$$

Conclusion:

We conclude that, the relationship between the finite measures and complex measures have been proved by the above theorems.

BIBLIOGRAPHY

1. **Topics in Algebra** - L.N. Herstein (second edition) John wiley and sons, 1975. Vikas publishing house (P) Ltd, New Delhi – 110002.
2. **Introduction to commutative Algebras –** B.F.Atiyah – I.G. Mcdonald.
3. **Graduate Texts Mathematics Associative Algebras** - Picard Spierce (Springer Verlen – Newyork – Berlin)
4. **Non-commutative Rings** - I.N. Herstein.
5. **Modern Algebra –** M.L. Santiage – Tata Mcgraw – Hill publishing company Limited – New Delhi (2001).